

Chapter 1

Linear Equations

1.1 Graphs of Linear Equations

An equation involving two variables, say \( x \) and \( y \), is simply a way of relating the values of those variables. When the \( x \) and \( y \) appear outside of exponents, roots, or other functions, we call the relationship linear. Another way to tell that a relationship is linear is to graph all pairs of points which make the equation true. If the resulting graph is a straight line, then the relationship is linear.

Example 1.1.1 Celsius and Fahrenheit Temperatures

The relationship between a Celsius temperature \( C \) and a Fahrenheit temperature \( F \) is given by the equation \( F = \frac{9}{5}C + 32 \). Determine if this is a linear relationship by plotting pairs of points satisfying the equation.

In the graph shown to the right, note that the line passes through the point \((0, 32)\), meaning that at \(0^\circ\) Celsius, it is \(32^\circ\) Fahrenheit. Also, the line passes through the point \((-17.7, 0)\), meaning that at \(0^\circ\) Fahrenheit, it is \(-17.7^\circ\) Celsius. A sampling of other points on the line is given below:

\[
\begin{array}{c|c}
C & F \\
10 & 50 \\
-10 & 14 \\
20 & 68 \\
-20 & -4 \\
\end{array}
\]

How then can we describe the relationship between Celsius and Fahrenheit temperature? The equation does a pretty good job. The line above does an even better job of depicting it graphically, but what does it really mean? Let’s examine the equation a bit more closely.

\[ F = \frac{9}{5}C + 32 \]

The Fahrenheit temperature is automatically 32 when \( C \) is 0, because of the +32 on the right had side of this equation. That is, the graph of the line will pass through the \( F - axis \) when \( F = 32 \). Because of this fact, and the tradition of calling the vertical axis the \( y \) axis, this number is often called the \( y \)-intercept. It gives us a starting place – a point – through which our graph must pass.

Finally, note that if the temperature increases \(1^\circ\) Celsius, because our variable \( C \) is multiplied by \(\frac{9}{5}\), we add \(\frac{9}{5}\) of a degree to \( F \). This is called the \(slope\) of the line shown above. It tells us how much \( F \) changes as \( C \) changes.
We could read off the slope, usually denoted $m$, and $y$-intercept, usually denoted as $b$, because the equation was in a very specific form. That form, called the “slope-intercept form” is given below.

$$ y = mx + b $$

Putting linear equations into this specific form allows us to easily find a single point on the graph, the $y$-intercept, $(0, b)$, and the slope of the graph.

**Example 1.1.2 Finding Slopes and Intercepts**

Find the slope and intercept which relate the variables $x$ and $y$ in the linear equations below.

$$ 3x + 2y = 7 $$

Our first move is to manipulate this equation until it is in slope-intercept form. We start by subtracting $3x$ from both sides to isolate the $y$ on the left hand side. This gives:

$$ 2y = -3x + 7 $$

Next, we need to divide by 2 so that $y$ is truly isolated. Doing this to both sides of the equation yields:

$$ y = \frac{-3}{2}x + \frac{7}{2} $$

So then, our slope, $m = \frac{-3}{2}$ and our intercept $b = \frac{7}{2}$. Thus, the graph of this line would pass through the point $(0, \frac{7}{2})$ and the $y$ values would increase $\frac{-3}{2}$ for every one unit increase in the $x$ value.

The original form of the equation above, $3x + 2y = 7$, is of particular interest. Notice that the $y$-intercept was $\frac{7}{2}$. That is, when we set the $x$ value to zero, the $y$ value is $\frac{7}{2}$. This makes sense! If $x = 0$ in this equation, then we have $0 + 2y = 7$ and we can solve for $y$ to get $\frac{7}{2}$. What do you suppose the $x$-intercept would be? If $y = 0$, the equation becomes $3x + 0 = 7$ and we can solve for $x$ to get $x = \frac{7}{3}$. It is often quicker to find the $x$ and $y$-intercepts and draw a straight line between them, than it is to graph in any other fashion.

**Example 1.1.3 Graphing a Line**

Graph the line $5x + 2y = 10$.

First, set $x = 0$ and solve for $y$:

$$ \begin{align*}
5(0) + 2y &= 10 \\
2y &= 10 \\
y &= 5
\end{align*} $$

Next, set $y = 0$ and solve for $y$:

$$ \begin{align*}
x + 2(0) &= 10 \\
x &= 10
\end{align*} $$

This gives us the two points $(0, 2)$ and $(5, 0)$ which are enough to determine the line.
Exercises

1. Find the slope and y-intercept of each of the following lines.
   
   (a) \( y = -3x + 1 \)
   (b) \( 2x + 6y = 4 \)
   (c) \( \frac{x-y}{2} = -1 \)

2. Graph each of the following lines using the method of setting \( x \) and \( y \) to zero.
   
   (a) \( 3x + 5y = 30 \)
   (b) \( 15x - 9y = 45 \)
   (c) \( -4x + 6y = 6 \)

3. Graph the following pair of lines using the method above, and find the point where they cross. What can you say about the \( x \) and \( y \) values of those points with respect to the two equations you graphed?
   
   \[
   x + 6y = 12 \\
   -2x + 4y = 8
   \]

4. Graph the following pair of lines using the method seen in this section. Is there a point where these lines cross? Find the slope of the two lines. What do you notice?
   
   \[
   x - 2y = -4 \\
   3x - y = -7
   \]
1.2 Systems of Equations

When two or more linear equations are looked at together, they are called a system of linear equations. One of the key questions in which we will be interested is when there are pairs of \((x, y)\) values which make all of the equations true.

**Example 1.2.1 System of Equations Solution**

Consider the two equations below:

\[
\begin{align*}
2x + 5y &= 13 \\
5x - y &= -8
\end{align*}
\]

The pair of values, \(x = -1\) and \(y = 3\) are a solution to this system of equations since on the one hand, \(2(-1) + 5(3) = -2 + 15 = 13\) and on the other hand, \(5(-1) - (3) = -5 - 3 = -8\).

In this case, there is only a single pair of \((x, y)\) values which makes both of these equations true. But is this always the way things work? One way we can investigate this further is to consider the ways two lines can interact graphically.

In the first figure, the slopes of the two lines are different, and as such, they cross in exactly one point. That means, there is a single \((x, y)\) point which makes both of the equations which go with these lines true. This is called a **unique solution** to the system of the equations.

The second figure shows two distinct lines which are parallel to one-another. Lines are parallel exactly when they have the same slope. In this case, the lines have the same slope, but clearly have different \(y\)-intercepts. Since the lines are parallel, they will never cross. Therefore, there is no pair of \((x, y)\) values which make both of the equations for these lines true at the same time. The system is said to have **no solution**.

In the final figure, the bold line represents two lines, one on top of the other. While this may seem contrived, suppose you were asked to find a solution to the two equations below:

\[
\begin{align*}
7x - 3y &= 17 \\
-14x + 6y &= -44
\end{align*}
\]

Notice that the second equation is exactly \(-2\) times the first equation. So then any pair of \((x, y)\) values which makes the first equation true, will automatically make the second equation true as well. So if we were to graph both equations, we would get two copies of the same line, one on top of the other. That means that there are **infinitely many solutions** to this system of equations. Note that it is still **not** the case that all \((x, y)\) pairs will solve the system—only those which are actually on the lines.
Example 1.2.2 Solution Types

Determine the type of solution set belonging to each of the following systems of equations.

1. \[4x + 7y = 28\]
   \[3x - 2y = 6\]

   We transform each equation into slope-intercept form, to determine the slopes.
   In the first case, we get: \[y = -\frac{4}{7}x + 7\], so the slope is \(-\frac{4}{7}\).
   And in the second case: \[y = \frac{3}{2}x - 3\], so the slope is \(\frac{3}{2}\).
   So there is a unique solution.

2. \[7y - 3x = 12\]
   \[7y - 3x = 4\]

   We transform each equation into slope-intercept form as before.
   In the first case, \(y = \frac{3}{7}x + \frac{12}{7}\), so the slope is \(\frac{3}{7}\) and the intercept is \(\frac{12}{7}\).
   In the second case, \(y = \frac{3}{7}x + \frac{4}{7}\), the slope is \(\frac{3}{7}\) and the intercept is \(\frac{4}{7}\).
   Since the slopes are the same, with different intercepts, no solutions exist.

3. \[3x + 2y = 14\]
   \[\frac{3}{2}x + y = 7\]

   We transform each equation into slope-intercept form again.
   In the first case, \(y = -\frac{3}{2}x + 7\).
   In the second case, \(y = -\frac{3}{2}x + 7\).
   Since the equations are identical, there are infinitely many solutions.

Now that we have seen the types of solutions that can arise, we need to discover how to find those solutions. We will look at two methods. The first, substitution, is easy to understand, but can sometimes be fairly involved. The second method, addition, is usually easy to apply, but takes some more explanation.

Substitution

The substitution method takes advantage of the fact that we know any solution to the system will make both equations true. Therefore, if we can take both equations and solve them for the same variable, we could set them equal to each other and eliminate one of the variables. The method below shows this procedure.

Example 1.2.3 Solution by Substitution

Solve the following system of equations using the substitution method.

\[
\begin{align*}
4x - 7y &= -5 \\
2x + 3y &= 17
\end{align*}
\]

Starting with the first equation, we solve for \(y\). This amounts to writing the first equation in the slope-intercept form. It yields \(y = \frac{4}{7}x + \frac{5}{7}\). Now, we do the same thing with the second equation. This gives \(y = -\frac{2}{3}x + \frac{17}{3}\). Now, as \(y\) must equal both of these, if it is to solve the system of equations, we can simply write \(\frac{4}{7}x + \frac{5}{7} = -\frac{2}{3}x + \frac{17}{3}\).

To simplify this, multiply everything by 21, to clear the fractions. This gives us the equation \(12x + 15 = -14x + 119\). Solving for \(x\) yields \(26x = 104\) so that \(x = 4\). Now, we take this value of \(x\) and substitute it back into either of the original equations. Choosing the first, we get \(4(4) - 7y = -5\) so that \(-7y = -21\) and thus \(y = 3\). So the solution to this system of equations is \((4, 3)\).

We actually did one more step in this process than was needed. Once we had solved the first equation for \(y\), it would have been enough to replace the \(y\) in the second equation with this first equality. That is, to substitute \(\frac{4}{7}x + \frac{5}{7}\) for \(y\) in the second equation.
Addition
The second method of solution also makes use of the fact that both equations are true at the same time. This time, however, we manipulate one of the equations to have exactly the opposite coefficient of either the $x$ or $y$ than that found in the second equation. This method is exhibited in the following example.

Example 1.2.4 Solution by Addition

Solve the following system of equations using the addition method.

$$
2x + 5y = 14
$$

$$
x - 3y = -4
$$

Notice that if we were to multiply everything in the second equation by -2, we would get an equation which is equivalent to the second equation, and also one which has the opposite coefficient of $x$ as that found in the first equation. If we go ahead and perform that multiplication, the system becomes:

$$
2x + 5y = 14
$$

$$
-2x + 6y = 8
$$

Now, if we add the right had sides of the equations and the left had sides of the equations together, we get the equation $2x - 2x + 5y + 6y = 14 + 8$. Simplifying, we get $11y = 22$. Therefore, $y = 2$. We can now substitute this $y$ value into either of the two equations and solve. Using the first equation, we get $2x + 5(2) = 14$, so that $2x + 10 = 14$ and therefore $2x = 4$. Hence, $x = 2$ and our solution is $(2, 2)$.

Exercises

1. In each of the following systems of equations, determine the type of solution set: unique solution, no solution, or infinitely many solutions.

   (a) $3x + 6y = 14$
       $2x + 4y = 25$

   (b) $12x + 15y = 35$
       $15x + 12y = 25$

   (c) $12x - 8y = 21$
       $36x - 24y = 63$

2. Each of the systems of equations below has a unique solution. Solve these systems by substitution.

   (a) $15x - 11y = 80$
       $10x + 5y = 115$

   (b) $4x + 7y = -1$
       $2x - y = -5$

   (c) $x + y = 12$
       $x - y = 8$

3. Solve the systems of equations in number 2 above by addition.
1.3 Graphing Regions

In the last section we looked at combining two or more linear equations into a system. A solution to that system is one or more pairs of points which make all equations in the system true. This concept can be extended to a system of linear inequalities in a similar fashion. Consider the following example.

**Example 1.3.1 System of Linear Inequalities**

Show that the point (3, 1) is a solution to the system of linear inequalities shown below.

\[
\begin{align*}
4x + 5y & \leq 20 \\
6x - 4y & \leq 24 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

As was the case with systems of linear equations, we need to verify that the point given, (3, 1), makes each of the inequalities true. Checking:

\[
\begin{align*}
4(3) + 5(1) &= 12 + 5 = 15 \leq 20 \\
6(3) - 4(1) &= 18 - 4 = 14 \leq 24 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

Hence, the point is a solution to the system.

So we now know how to check to see that a point is a solution to a system of linear inequalities. But how do we come up with a solution set? We have to resort to graphing again, as that is the best way to keep track of all of the linear inequalities.

**Example 1.3.2 Graphing Systems of Inequalities**

Graph the four linear inequalities above, and shade in the region which contains points when make all of the inequalities true.

If we graph the first inequality and check the point (0, 0) we find that everything below the line is shaded. If we do the same with the second line, we find everything above the line is shaded. Finally, the two single-variable inequalities tell us that we must shade above and to the right of the axis.

Now combining these shadings, we look for the area which is shaded by all four inequalities. This region is the solution set.
Our next step is to try to find the corners of this region. This will give a more concrete description of the region which contains the solutions to our system of linear equations. While the graph is helpful here in determining where we should look for corners, the basic method we will use is solving systems of equations.

**Example 1.3.3 Finding Corner Points**

Find the corner points of the solution region found in the previous example.

There will be four corners. Three of them are easy to find. The two inequalities \( x \geq 0 \) and \( y \geq 0 \) both go with the lines \( x = 0 \) and \( y = 0 \). The point which makes both of these true is \((0, 0)\).

The second corner we will examine is where the lines \( y = 0 \) and \( 6x - 4y = 24 \) meet. To find this, we solve the system of equations:

\[
\begin{align*}
y &= 0 \\ 6x - 4y &= 24
\end{align*}
\]

This system is ready to solve by substitution. We simply replace the \( y \) in the second equation with 0 and solve. Then, \( 6x - 4(0) = 24 \) so that \( 6x = 24 \) and finally, \( x = 4 \). Thus, the solution is \((4, 0)\).

The third corner is found when the lines \( x = 0 \) and \( 4x + 5y = 20 \) cross. This is another system of linear equations.

\[
\begin{align*}
x &= 0 \\ 4x + 5y &= 20
\end{align*}
\]

Again, substitution is probably the easiest method to use here. We can replace the \( x \) with 0 and solve. Doing this, we see that \( 4(0) + 5y = 20 \) and thus \( 5y = 20 \) giving \( y = 4 \). Hence, the solution and corner point is \((0, 4)\).

Finally, the fourth corner point occurs when the lines \( 4x + 5y = 20 \) and \( 6x - 4y = 24 \) meet. This is again a system of equations, and is given below.

\[
\begin{align*}
4x + 5y &= 20 \\ 6x - 4y &= 24
\end{align*}
\]

This system is probably best solved by addition. If we multiply the first equation by 3 and the second equation by -2, we get the equivalent system:

\[
\begin{align*}
12x + 15y &= 60 \\ -12x + 8y &= -48
\end{align*}
\]

Adding these together, we see that \( 23y = 12 \) so that \( y = \frac{12}{23} \). While this isn’t necessarily a “nice” solution, we can plug it in for \( y \) in one of the two equations above, we will use the first equation, and solve for \( x \). Doing this, we see that \( 4x + 5 \left( \frac{12}{23} \right) = 20 \) and then \( 4x + \frac{60}{23} = 20 \). Then \( 4x = \frac{400}{23} \) and finally, \( y = \frac{100}{23} \). This gives us the final corner point, \( \left( \frac{12}{23}, \frac{100}{23} \right) \).
Exercises

1. Graph the solution region for each of the systems of inequalities found below.

   (a) \[ 3x - 6y \leq 4 \]
       \[ y \geq 0 \]
       \[ x \geq 0 \]

   (b) \[ 2x + 8y \leq 16 \]
       \[ 4x + 3y \leq 12 \]
       \[ x \geq 0 \]
       \[ y \geq 0 \]

   (c) \[ x - y \geq 2 \]
       \[ 2x + 3y \leq 12 \]
       \[ 8x + 5y \leq 12 \]
       \[ x \geq 0 \]

2. Find all corner points of each of the regions graphed in the previous problem.
1.4 Systems of Linear Equations from Story Problems

We now turn to applications of systems of linear equations. They are more than just curious mathematical expressions which produce meaningless lines on a graph. Linear equations arise in every day life, and systems of linear equations are themselves quite common and important for solving real-world problems. Consider the following example.

Example 1.4.1 A Music Store

You go to a music store to buy CD’s and Tapes. The CD’s cost $12 each and the tapes cost $7 each. You buy 14 items, spending a total of $123. How many tapes and CD’s did you buy?

1. The first step in solving any story problem like this is to clearly identify variables. In our case, we are asked how many tapes and CD’s were purchased. These are the unknowns, so we make the definitions:

   \[ x = \text{number of CD’s} \quad y = \text{number of Tapes} \]

2. Our next step is to set-up a system of equations. For this, it is often a good idea to set up a table. The columns in the table go with the variables while the rows in the table represent the various resources or pieces of information given. In this case, our resources are number of items bought (we leave the store with 14 items) and money (we spend $123). So, our table will look like the following:

<table>
<thead>
<tr>
<th></th>
<th>CD’s</th>
<th>Tapes</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Items</strong></td>
<td>12x</td>
<td>+ 7y</td>
<td>= 14</td>
</tr>
<tr>
<td><strong>Money</strong></td>
<td></td>
<td></td>
<td>= 123</td>
</tr>
</tbody>
</table>

This table actually contains the two equations we will use in the next step to solve the problem.

3. Finally, we have reduced this problem to a purely algebraic question. Namely, what pair of \((x, y)\) values makes the following system of equations true:

   \[
   \begin{align*}
   x + y &= 14 \\
   12x + 7y &= 123
   \end{align*}
   \]

   We can solve this system using either substitution or addition. Substitution works out nicely if we solve the first equation for \(y\) and substitute into the second equation.

   \[
   y = 14 - x \quad \Rightarrow \quad 12x + 7(14 - x) = 123 \quad \Rightarrow \quad 12x + 98 - 7x = 123
   \]

   \[
   5x = 25 \quad \Rightarrow \quad x = 5 \quad \Rightarrow \quad 5 + y = 14 \quad \Rightarrow \quad y = 9
   \]

4. The last step is to state the solution in an intelligible manner. This is where we refer back to our variable definition, and to our numbers found above, and come up with the statement: “We buy 5 CD’s and 9 tapes.”

While the basic steps are always the same in solving story problems such as these, a good deal of the challenge is in interpreting different situations correctly. That is, we always have to identify the variables (unknowns) and the equations (resources) which go into our setup. In the next few examples we will dispense with the step-by-step commentary, but will still take the time to identify these important items in each problem.
Example 1.4.2 Pizza Production

A pizza restaurant makes two sizes of cheese pizza—large and small. The large pizzas requires $\frac{3}{2}$ lb. of pizza dough and $1 \frac{1}{2}$ lbs. of cheese each. The small pizzas each use $\frac{1}{2}$ lb. of dough and $\frac{1}{2}$ lb. of cheese. At the end of the week there are 4 lbs. of dough and 11.5 lbs. of cheese left to use up before it goes bad. How many of each pizza size should the restaurant make to use up all of these ingredients.

\[ x = \text{number of large pizzas made} \quad y = \text{number of small pizzas made} \]

Based on the problem, our resources are dough and cheese. So our columns in the table will be number of large and small pizzas, while our rows will be dough and cheese. In each cell we place the amount of resource required per item. This produces the following table.

<table>
<thead>
<tr>
<th></th>
<th>Large</th>
<th>Small</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dough</td>
<td>$\frac{3}{2}x$</td>
<td>$\frac{1}{4}y$</td>
<td>4</td>
</tr>
<tr>
<td>Cheese</td>
<td>$\frac{1}{2}x$</td>
<td>$\frac{1}{2}y$</td>
<td>$\frac{23}{2}$</td>
</tr>
</tbody>
</table>

Thus, we wish to solve the system of equations above. The fractions are somewhat distracting, so it is probably a good idea to try to get rid of them. We can do this by multiplying the top equation by 4, and the bottom equation by 2. This yields:

\[ 2x + y = 16 \]
\[ 3x + 2y = 23 \]

Solving by either addition or substitution yields the values $x = 7$ and $y = 2$. Thus, our solution is: “The restaurant should make 7 large pizzas and 2 small pizzas.”

So far, we have been working with linear equations with two variables. Both of our examples in this section have involved only two variables. However, our techniques will extend without problem to three variable problems, or even larger problems. Consider the following three-variable problem.

Example 1.4.3 Toy Production

A certain toy manufacturer makes 3 kinds of toys. The toy doll requires 8 oz. of plastic, 2 oz. of metal, and 15 in$^2$ of fabric. The toy truck requires 2 oz. of plastic and 16 oz. of metal. The toy action figure requires 6 oz of plastic and 8 in$^2$ of fabric. how many of each type of toy should be manufactured to completely use up 164 oz. of plastic, 254 oz. of metal, and 209 in$^2$ of fabric?

\[ x = \# \text{ dolls} \quad y = \# \text{ trucks} \quad z = \# \text{ action figures} \]

Our variables are given above, so our resources are the three items left: plastic, metal, and fabric. This gives us the following table.

<table>
<thead>
<tr>
<th></th>
<th>Dolls</th>
<th>Trucks</th>
<th>Figures</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plastic</td>
<td>$8x$</td>
<td>$2y$</td>
<td>$6z$</td>
<td>164</td>
</tr>
<tr>
<td>Metal</td>
<td>$2x$</td>
<td>$16y$</td>
<td>$0z$</td>
<td>254</td>
</tr>
<tr>
<td>Fabric</td>
<td>$15x$</td>
<td>$0y$</td>
<td>$8z$</td>
<td>209</td>
</tr>
</tbody>
</table>

In this situation, actually solving the problem is somewhat more difficult. Probably the best method is to use substitution. Since the second and third equations have 0’s as coefficients, they are the best places to start. From the second equation, we see that $y = \frac{127}{8} - \frac{x}{4}$. From the third equation, we see that $z = \frac{209}{8} - \frac{15}{8}x$. Substituting these both into the first equation allows us to solve for $x$ since we then have $8x + 2\left(\frac{127}{8} - \frac{x}{4}\right) + 6\left(\frac{209}{8} - \frac{15}{8}x\right) = 164$. So then $8x + \frac{127}{4} - \frac{x}{2} + \frac{627}{4} - \frac{45}{4}x = 164$ and thus $8x - \frac{46}{4}x = 164 - \frac{377}{2}$ giving $-\frac{1}{2}x = \frac{49}{2}$ and hence $x = 7$. Further work yields $y = 15$ and $z = 13$.

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Exercises

1. You have $10,000 to invest in CD’s and bonds. You buy a CD which earns 5.5% interest a year, and a bond that earns 8% interest a year. How much should you invest in each if you want to earn the same amount from the CD’s and bonds during the first year.

2. Herbert and Gertrude plan to open a fruit-drink stand. They have 45 lemons and 30 oranges to use in making two types of drinks. They use 2 lemons and 1 orange to make 10 glasses of the tart drink and 1 lemon and 3 oranges to make 10 glasses of the sweet drink. Find the number of glasses of each type of drink they should make to use exactly their supply of fruit.

3. Murphy’s Muffin Shoppe makes both large and small bran muffins. Each large muffin uses 4 ounces of dough and 2 ounces of bran, while a small muffin uses 1 ounce of dough and 1 ounce of bran. Each day there are 300 ounces of dough and 160 ounces of bran available. How many large and small muffins should be made each day to use up all of the available dough and bran?

4. A bakery makes three types of desserts: chocolate taffy, cheese cake, and fudge. At the end of the day, the baker must use up 44 pounds of sugar, 53 pounds of chocolate, and 35 pounds of butter to make room for the next morning’s supply delivery. Each batch of taffy only requires one pound of each ingredient. A batch of cheese cake requires 4 pounds of sugar, 5 pounds of chocolate, and 3 pounds of butter. Finally, a batch of fudge takes 2 pounds of sugar, 3 pounds of chocolate, and 1 pound of butter. Assuming that she has plenty of other ingredients, how many batches of each should the baker make to use up all of her sugar, chocolate, and butter?
1.5 Linear Programming Set-up

The systems of equations we have just looked at are ones in which we are looking for a unique solution. That is, we are looking for the way to use up all of our dough and cheese exactly by making small and large pizzas. This situation, while sometimes practical, is not the usual sort of real-world problem.

What if we did not care about using up exactly the resources which are on hand, but rather we just wanted to make as much profit as possible, or reduce costs as much as possible? This situation is much more common, since there is typically no requirement that all of a certain resource be used.

Consider now a restaurant making two types of salsa: mild and hot. Suppose that the mild salsa brings in $3 per batch, and a batch of hot salsa brings in $7 in profit. Given a limited number of resources, how would you use them? Making all hot salsa seems like it might be the logical course of action. After all, the restaurant makes $4 more per batch! But is this really the case? Maybe not! Perhaps we can make more profit by selling a few batches of mild salsa as well to better utilize our resources!. In this section, we will set-up this sort of problem. Solving will take place in the next section.

Example 1.5.1 Mexican Salsa Sales

A Mexican restaurant sells two kinds of salsa—hot and mild. The main ingredients in each are tomatoes and jalapenos. The restaurant has 100 pounds of jalapenos and 400 pounds of tomatoes on hand. It takes one pound of jalapenos and 10 pounds of tomatoes to make a batch of mild salsa. A batch of hot salsa requires 3 pounds of jalapenos and 8 pounds of tomatoes. If the restaurant makes $3 profit on each batch of mild salsa, and $7 profit on each batch of hot salsa, how many batches of each should be made to maximize profit?

We begin just as we did in the previous section, by identifying unknowns and resources. In this case, the unknowns are the number of batches of each type of salsa which should be made.

\[ x = \text{ number of batches of mild } \quad y = \text{ number of batches of hot} \]

Our resources are jalapenos and tomatoes. So our columns in the table will be number of batches of each type of salsa, while our rows will be tomatoes and jalapenos. In each cell we again place the amount of resource required per batch made. This produces the following table.

<table>
<thead>
<tr>
<th></th>
<th>Mild</th>
<th>Hot</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tomatoes</td>
<td>10x</td>
<td>8y</td>
<td>≤ 400</td>
</tr>
<tr>
<td>Jalapenos</td>
<td>1x</td>
<td>3y</td>
<td>≤ 100</td>
</tr>
</tbody>
</table>

In this case, we use the inequality ≤ because we do not need to use up exactly 400 tomatoes or 100 jalapenos. Rather, we can use no more than 400 tomatoes and 100 jalapenos. In addition, there are two “common sense” inequalities which should be added. They are \( x \geq 0 \) and \( y \geq 0 \), stemming from the fact that we can not make less than zero batches of salsa! This gives us a set of constraints, which in the next section will help us define a “feasible set” of possible solutions.

Constraints:

\[
\begin{align*}
10x + 8y & \leq 400 \\
x + 4y & \leq 100 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

There is one new piece of information we need to add to this set-up. Namely, we need to take into account the amount of money we make off of each sale. Examining the problem statement, we see that we make $3 for every batch of mild salsa \((x)\) and $7 for every batch of hot \((y)\). Hence, our objective is:

**Objective Function:** Maximize \( P = 3x + 7y \)

This method gives us the basic information we will use to solve linear programming problems such as the one posed here. The actual solution will have to wait for a few pages.
Example 1.5.2 Cruise Line

Columbus cruise-line offers one-week cruises on their three ships: The Nina, Pinta, and Santa Maria. The Nina has 500 regular cabins and 200 deluxe cabins. The Pinta has 400 regular and 400 deluxe cabins. And the Santa Maria has 800 regular and 500 deluxe cabins. The cost to run each of the ships is $100,000 per week for the Nina, $120,000 per week for the Pinta, and $180,000 per week for the Santa Maria. There is a demand for 12,000 one-week regular cruises and 8,000 one-week deluxe cruises during the cruise season. How many weeks should each ship be scheduled to meet the demand at a minimum cost.

First, note that there are three unknowns in this case. These are the number of weeks each cruise ship sails.

\[ x = \text{# weeks N sails} \quad y = \text{# weeks P sails} \quad z = \text{# weeks S sails} \]

Our resources are the two types of rooms: deluxe and regular. In this case, we need to meet a given demand, so we need to provide at least 12,000 regular cabins and 8,000 deluxe cabins. This means that resource equations/inequalities will actually use the \( \geq \) inequality. The table is shown below.

<table>
<thead>
<tr>
<th></th>
<th>Nina</th>
<th>Pinta</th>
<th>Santa Maria</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular</td>
<td>500x</td>
<td>400y</td>
<td>800z</td>
<td>( \geq 12,000 )</td>
</tr>
<tr>
<td>Deluxe</td>
<td>200x</td>
<td>400y</td>
<td>500z</td>
<td>( \geq 8,000 )</td>
</tr>
</tbody>
</table>

This problem asks us to minimize costs. So, we need to determine what the cost function is. If \( x \) is the number of weeks that the Nina sails, then the total amount of money that we spend on sailing this ship is $100,000x. Continuing with this, our cost formula is that found below, and our final set-up is:

**Objective Function:** Minimize \( C = 100000x + 120000y + 180000z \)

**Subject To the Constraints:**

\[
\begin{align*}
500x + 400y + 800z & \geq 12000 \\
200x + 400y + 500z & \geq 8000
\end{align*}
\]

\[ x \geq 0 \]

\[ y \geq 0 \]

\[ z \geq 0 \]

**Exercises:** Set-up but do not solve the following linear programming problems.

1. A roadside drink stand sells two types of citrus drink: tart and sweet drink. They charge $0.75 per glass of tart drink and $1.00 per glass of sweet drink. It takes 2 lemons and 1 orange to make a glass of tart drink, and 3 oranges to make a glass of sweet drink. If they have 50 lemons and 75 oranges, how many glasses of each sort of drink should the stand make to maximize income?

2. An internet company has two types of servers. The first server can provide for 50 web sites and 1500 email addresses. The second type of server can host 25 web sites and 3500 email addresses. It costs $3,500 to purchase the first type of server, and $2,750 to purchase the second type of server. If there is a demand for the hosting of 500 web sites and 15,000 email addresses, how many of each type of server should be purchased to meet this demand while minimizing costs?

3. A florist sells three types of flower arrangements: small, medium, and large. The large bouquets require 10 roses, 15 daisies and 25 carnations. The medium arrangements use 8 roses, 5 daisies, and 15 carnations. The small arrangement uses only 4 roses, 4 daisies, and 4 carnations. The florist charges $75 per large, $60 per medium, and $45 per small arrangement. In a given week, the florist receives 100 roses, 150 daisies and 250 carnations from the greenhouse. How many of each arrangement should the florist make (and presumably sell) to maximize profit?
1.6 Solving Linear Programming Problems

We are now ready to solve problems involving constraints and objective functions. These problems are called linear programming problems because they involve linear equations and inequalities. We have done most of the set-up work in the previous sections—all that remains is to put it together. We do this in the next example.

Example 1.6.1 Hotel Management

A company has two hotels, call them A and B. Hotel A has 10 single rooms and 12 double rooms per floor, and is 10 stories tall. It costs $10,000 per floor per night to operate hotel A. Hotel B is larger with 15 floors, each having 12 single and 8 double rooms. It costs $8,500 per floor per night to operate hotel B. If there is a demand for 100 single rooms and 80 double rooms per night in the areas in which these hotels are located, how many floors in each hotel should be opened to minimize operating cost while still meeting demand?

Using the techniques from the last section, we can set up the variables, constraints, and objective function as follows.

Unknowns:
\[ x = \text{number of floors in hotel A to open} \]
\[ y = \text{number of floors in hotel B to open} \]

Objective Function:
\[ \text{Minimize Cost} = 10000x + 8500y \]

Constraints:
\[ 10x + 12y \geq 100 \]
\[ 12x + 8y \geq 80 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ x \leq 10 \]
\[ y \leq 15 \]

Graphing the region defined by these constraints and finding the corner points we get the picture shown above. Note that most of the corner points are very easy to find. However, we need to solve the system of equations defined by the first two inequalities (changed into equations) to find the last corner point. These points are: (10, 0), (10, 15), (0, 15), (0, 10), and \( \left( \frac{5}{2}, \frac{25}{4} \right) \).

One of the nice things about linear equations is that they will have their maximum or minimum on one of the corner points of the feasible set graphed above. So to determine how to get the lowest cost, we just need to check our objective function at each corner point. This is done in the table below.

<table>
<thead>
<tr>
<th>Point</th>
<th>Floors in A</th>
<th>Floors in B</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 0)</td>
<td>10</td>
<td>0</td>
<td>10,000(10) + 8,500(0) = 100,000</td>
</tr>
<tr>
<td>(10, 15)</td>
<td>10</td>
<td>15</td>
<td>10,000(10) + 8,500(15) = 227,500</td>
</tr>
<tr>
<td>(0, 15)</td>
<td>0</td>
<td>15</td>
<td>10,000(0) + 8,500(15) = 127,500</td>
</tr>
<tr>
<td>(0, 10)</td>
<td>0</td>
<td>10</td>
<td>10,000(0) + 8,500(10) = 85,000</td>
</tr>
<tr>
<td>( \left( \frac{5}{2}, \frac{25}{4} \right) )</td>
<td>( \frac{5}{2} )</td>
<td>( \frac{25}{4} )</td>
<td>10,000(( \frac{5}{2} )) + 8,500(( \frac{25}{4} )) = 78,125</td>
</tr>
<tr>
<td>( \left( \frac{5}{2}, \frac{25}{4} \right) )</td>
<td>3</td>
<td>( \frac{7}{2} )</td>
<td>10,000(3) + 8,500(7) = 89,500</td>
</tr>
</tbody>
</table>

The minimum happens when there are 2.5 floors open in hotel A and 6.25 floors open in hotel B. However, if we allow only whole floors to be opened at a time, we would be forced to bump this corner up to (3, 7) in which case it would no longer be the minimum, and it would be better to open all of hotel B, and close hotel A entirely!
Recall that we saw some feasible sets (regions on the graph) earlier which were called unbounded because they could not be contained in a circle. That is, some of their edges did not have corner points. What do we do in a situation where we need to minimize or maximize an objective function on such a region? In the next example, we will explore that question.

**Example 1.6.2 Unbounded Feasible Sets**

Maximize and minimize the function $2x - y$ subject to the constraints: $2x + 3y \geq 6$, $3x - 2y \geq -6$, $x \geq 1$ and $y \geq 1$.

Steps to solve:

1. Graph the constraints to get feasible set.
2. Find all corner points of feasible set.
3. Check value of objective function on corners.
4. Check further along any edges going to $\infty$.

Notice that in the figure to the left two of the edges keep going on to infinity. We will need to check a test point further along those edges to see if the objective function is getting larger or smaller.

As before, we place all of our corner points into a table and check the value of the objective function at each point. The corner points are found by determining where the various lines cross.

<table>
<thead>
<tr>
<th>Point</th>
<th>Objective Function $2x - y$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, \frac{7}{3})$</td>
<td>$2(1) - \frac{7}{3} = \frac{1}{3}$</td>
<td>Maximum?</td>
</tr>
<tr>
<td>$(\frac{3}{2}, 1)$</td>
<td>$2(\frac{3}{2}) - 1 = 2$</td>
<td>Minimum?</td>
</tr>
<tr>
<td>$(1, \frac{9}{2})$</td>
<td>$2(1) - \frac{9}{2} = -\frac{5}{2}$</td>
<td></td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$2(2) - 1 = 3$</td>
<td>Larger, so no Maximum</td>
</tr>
<tr>
<td>$(2, 6)$</td>
<td>$2(2) - 6 = -2$</td>
<td>Larger, so above is Minimum</td>
</tr>
</tbody>
</table>

In the table above we first check the three corners given. This seems to indicate that the point $(\frac{3}{2}, 1)$ is a maximum for the objective function. However, since this point comes from one of the region boundaries which keeps going to infinity (the line $y = 1$ in this case) we need to pick a point further along that line. We choose the point $(2, 1)$ and check its value. The objective function is 3 at that point, which is larger than the supposed minimum 2. This means that the objective function will keep getting larger as we move along this line, so there is no maximum on this feasible set!

The point $(1, \frac{9}{2})$ appears to be a minimum for the feasible set. Again, it is on the line $3x - 2y = -6$ which is a border of the feasible set extending to infinity. We need to check a point further along this line! We choose the $x$ value 2, plug it in to the equation, and find that $3(2) - 2y = -6$ gives $-2y = -12$ so that $y = 6$. Hence, we test the point $(2, 6)$. This point yields a value of $-2$ which is larger than the potential minimum. So the objective function is getting larger as we move along this line, meaning that our original point $(1, \frac{9}{2})$ is indeed the minimum!

The procedure followed for solving this problem is just like that followed in the first example, with the exception of the extra points which need to be checked. Whenever you are looking for a minimum or maximum on some region, you should ask yourself if the region is bounded. If you can draw a circle around it, then you can just check the corner points and be sure of your results. If you can not draw a circle around the region—if it keeps going off to infinity—you may have to check extra points if your potential minimum or maximum is a point on one of the boundary lines which goes to infinity.
Exercises

1. Find the minimum and maximum of the objective function $2x - 5y$ subject to the constraints $x + 3y \leq 30$, $3x - 4y \geq -27$, $x \geq 0$, and $y \geq 0$.

2. Find the maximum of $2x - 6y$ subject to the constrains $x + y \geq 4$, $4x - y \geq 1$, and $x - 4y \leq 0$.

3. Sam’s Deli makes sandwiches using bread and cheese. Each regular sandwich uses 6 inches of bread and 2 slices of cheese. Each large sandwich uses 10 inches of bread and 4 slices of cheese. The profit on a regular sandwich is $0.80 and the profit on a large sandwich is $1.20. Each day the deli has 110 feet of bread and 480 slices of cheese available. How many sandwiches of each size should be made to maximize profit?

4. Brown Brothers Box Company produces both standard and heavy-duty shipping containers. One standard container requires 1 square foot of 100-pound test cardboard and 3 square feet of liner board. One heavy-duty container requires 5 square feet of 100-pound test cardboard and 1 square foot of liner board. Each week the company has 5000 square feet of 100-pound test cardboard and 4500 square feet of linear board available. There is a commitment to produces at least 500 standard containers each week. If the cost of materials and labor is $0.30 for each standard container and $0.40 for each heavy-duty container, how many of each type should be produced to minimize these costs?