MATH 105: Finite Mathematics
7-4: Conditional Probability

Prof. Jonathan Duncan

Walla Walla College

Winter Quarter, 2006
Outline

1. Introduction to Conditional Probability
2. Some Examples
3. A “New” Multiplication Rule
4. Conclusion
Introduction to Conditional Probability

Some Examples

A “New” Multiplication Rule

Conclusion
In 1991 the following problem caused quite a stir in the world of mathematics.

Monty Hall Problem

Monty Hall, the host of “Let’s Make a Deal” invites you to play a game. He presents you with three doors and tells you that two of the doors hide goats, and one hides a new car. You get to choose one door and keep whatever is behind that door.

You choose a door, and Monte opens one of the other two doors to reveal a goat. He then asks you if you wish to keep your original door, or switch to the other door?

Play the Game
Monty Hall Solution

You should switch doors.

- You choose Door A and have a $\frac{1}{3}$ probability of winning.
- Monty eliminates a goat behind one of the other doors.
- Switching wins in cases 1 and 2 and looses in case 3.
- Thus, switching raises your probability of winning to $\frac{2}{3}$. 
Monty Hall Solution

You should switch doors.

<table>
<thead>
<tr>
<th></th>
<th>Door A</th>
<th>Door B</th>
<th>Door C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>goat</td>
<td>goat</td>
<td>car</td>
</tr>
<tr>
<td>2</td>
<td>goat</td>
<td>car</td>
<td>goat</td>
</tr>
<tr>
<td>3</td>
<td>car</td>
<td>goat</td>
<td>goat</td>
</tr>
</tbody>
</table>

Example:
- You choose Door A and have a \( \frac{1}{3} \) probability of winning.
- Monty eliminates a goat behind one of the other doors.
- Switching wins in cases 1 and 2 and loses in case 3.
- Thus, switching raises your probability of winning to \( \frac{2}{3} \).
**Monty Hall Solution**

You should switch doors.

<table>
<thead>
<tr>
<th></th>
<th>Door A</th>
<th>Door B</th>
<th>Door C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>goat</td>
<td>goat</td>
<td>car</td>
</tr>
<tr>
<td>2</td>
<td>goat</td>
<td>car</td>
<td>goat</td>
</tr>
<tr>
<td>3</td>
<td>car</td>
<td>goat</td>
<td>goat</td>
</tr>
</tbody>
</table>

Example:

- You choose Door A and have a $\frac{1}{3}$ probability of winning.
- Monty eliminates a goat behind one of the other doors.
- Switching wins in cases 1 and 2 and looses in case 3.
- Thus, switching raises your probability of winning to $\frac{2}{3}$. 
Monty Hall Solution

You should switch doors.

<table>
<thead>
<tr>
<th></th>
<th>Door A</th>
<th>Door B</th>
<th>Door C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>goat</td>
<td>goat</td>
<td>car</td>
</tr>
<tr>
<td>2</td>
<td>goat</td>
<td>car</td>
<td>goat</td>
</tr>
<tr>
<td>3</td>
<td>car</td>
<td>goat</td>
<td>goat</td>
</tr>
</tbody>
</table>

Example:

- You choose Door A and have a $\frac{1}{3}$ probability of winning.
- Monty eliminates a goat behind one of the other doors.
- Switching wins in cases 1 and 2 and loose in case 3.
- Thus, switching raises your probability of winning to $\frac{2}{3}$. 
Monty Hall Solution

You should switch doors.

<table>
<thead>
<tr>
<th></th>
<th>Door A</th>
<th>Door B</th>
<th>Door C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>goat</td>
<td>goat</td>
<td>car</td>
</tr>
<tr>
<td>2</td>
<td>goat</td>
<td>car</td>
<td>goat</td>
</tr>
<tr>
<td>3</td>
<td>car</td>
<td>goat</td>
<td>goat</td>
</tr>
</tbody>
</table>

Example:

- You choose Door A and have a $\frac{1}{3}$ probability of winning.
- Monty eliminates a goat behind one of the other doors.
- Switching wins in cases 1 and 2 and loses in case 3.
- Thus, switching raises your probability of winning to $\frac{2}{3}$. 
You should switch doors.

<table>
<thead>
<tr>
<th></th>
<th>Door A</th>
<th>Door B</th>
<th>Door C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>goat</td>
<td>goat</td>
<td>car</td>
</tr>
<tr>
<td>2</td>
<td>goat</td>
<td>car</td>
<td>goat</td>
</tr>
<tr>
<td>3</td>
<td>car</td>
<td>goat</td>
<td>goat</td>
</tr>
</tbody>
</table>

Example:

- You choose Door A and have a $\frac{1}{3}$ probability of winning.
- Monty eliminates a goat behind one of the other doors.
- Switching wins in cases 1 and 2 and loses in case 3.
- Thus, switching raises your probability of winning to $\frac{2}{3}$. 
Here is another example of **Conditional Probability**.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.</td>
</tr>
<tr>
<td>1. What is the probability that both are red?</td>
</tr>
<tr>
<td>2. What is the probability that both are red given that the first is white?</td>
</tr>
<tr>
<td>3. What is the probability that both are red given that the first is red?</td>
</tr>
</tbody>
</table>
Conditional Probability

Here is another example of Conditional Probability.

Example

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red?
2. What is the probability that both are red given that the first is white?
3. What is the probability that both are red given that the first is red?
Conditional Probability

Here is another example of **Conditional Probability**.

**Example**

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red?
2. What is the probability that both are red given that the first is white?
3. What is the probability that both are red given that the first is red?
Here is another example of **Conditional Probability**.

**Example**

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red?
   
   \[
   \frac{C(8, 2)}{C(10, 2)} = \frac{28}{45}
   \]

2. What is the probability that both are red given that the first is white?

3. What is the probability that both are red given that the first is red?
Here is another example of **Conditional Probability**.

**Example**

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red? \( \left( \frac{28}{45} \right) \)
2. What is the probability that both are red given that the first is white?
3. What is the probability that both are red given that the first is red?
Conditional Probability

Here is another example of **Conditional Probability**.

**Example**

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red? \( \frac{28}{45} \)
2. What is the probability that both are red **given** that the first is white?
   This can’t happen
3. What is the probability that both are red **given** that the first is red?
Conditional Probability

Here is another example of **Conditional Probability**.

**Example**

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red? \( \frac{28}{45} \)
2. What is the probability that both are red **given** that the first is white? (0)
3. What is the probability that both are red **given** that the first is red?
Here is another example of **Conditional Probability**.

**Example**

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red? \( \frac{28}{45} \)
2. What is the probability that both are red given that the first is white? (0)
3. What is the probability that both are red given that the first is red? \( \frac{7}{9} \)
Conditional Probability

Here is another example of **Conditional Probability**.

**Example**

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red? \( \frac{28}{45} \)

2. What is the probability that both are red given that the first is white? (0)

3. What is the probability that both are red given that the first is red?

In the last two questions, extra information changed the probability.
Conditional Probability

Information given about one event can effect the probability of a second event. Knowing that the first ball was white in the problem above changed the probability that both balls were red.

If $A$ and $B$ are events in a sample space then the probability of $A$ happening given that $B$ happens is denoted $\Pr[A \mid B]$ which is read “The probability of $A$ given $B$.”
Conditional Probability

Information given about one event can effect the probability of a second event. Knowing that the first ball was white in the problem above changed the probability that both balls were red.

Conditional Probability

If $A$ and $B$ are events in a sample space then the probability of $A$ happening given that $B$ happens is denoted

$$\Pr[A \mid B]$$

which is read “The probability of $A$ given $B$.”
Venn Diagrams and Conditional Probability

To help us develop a formula for $\Pr[A|B]$ we will use Venn Diagrams in the following example.

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>The probability of an event $A$ is 0.50. The probability of an event $B$ is 0.70. The probability of $A \cap B$ is 0.30. Find $\Pr[A</td>
</tr>
</tbody>
</table>

\[
\Pr[A|B] = \frac{0.30}{0.40 + 0.30} \approx 0.43
\]

\[
\Pr[B|A] = \frac{0.30}{0.30 + 0.20} \approx 0.50
\]
Venn Diagrams and Conditional Probability

To help us develop a formula for $\Pr[A|B]$ we will use Venn Diagrams in the following example.

**Example**

The probability of an event $A$ is 0.50. The probability of an event $B$ is 0.70. The probability of $A \cap B$ is 0.30. Find $\Pr[A|B]$ and $\Pr[B|A]$.

\[
\Pr[A|B] = \frac{0.30}{0.40 + 0.30} \approx 0.43
\]

\[
\Pr[B|A] = \frac{0.30}{0.30 + 0.20} \approx 0.60
\]
Venn Diagrams and Conditional Probability

To help us develop a formula for \( \Pr[A|B] \) we will use Venn Diagrams in the following example.

**Example**

The probability of an event \( A \) is 0.50. The probability of an event \( B \) is 0.70. The probability of \( A \cap B \) is 0.30. Find \( \Pr[A|B] \) and \( \Pr[B|A] \).

\[
\Pr[A|B] = \frac{0.30}{0.40 + 0.30} \approx 0.43
\]

\[
\Pr[B|A] = \frac{0.30}{0.30 + 0.20} \approx 0.60
\]
To help us develop a formula for $\text{Pr}[A|B]$ we will use Venn Diagrams in the following example.

**Example**

The probability of an event $A$ is 0.50. The probability of an event $B$ is 0.70. The probability of $A \cap B$ is 0.30. Find $\text{Pr}[A|B]$ and $\text{Pr}[B|A]$.

\[
\text{Pr}[A|B] = \frac{0.30}{0.40 + 0.30} \approx 0.43
\]

\[
\text{Pr}[B|A] = \frac{0.30}{0.30 + 0.20} \approx 0.60
\]
Venn Diagrams and Conditional Probability

To help us develop a formula for $\text{Pr}[A|B]$ we will use Venn Diagrams in the following example.

**Example**

The probability of an event $A$ is 0.50. The probability of an event $B$ is 0.70. The probability of $A \cap B$ is 0.30. Find $\text{Pr}[A|B]$ and $\text{Pr}[B|A]$.

$$
\text{Pr}[A|B] = \frac{0.30}{0.40 + 0.30} \approx 0.43
$$

$$
\text{Pr}[B|A] = \frac{0.30}{0.30 + 0.20} \approx 0.60
$$
Conditional Probability Formula

Let $A$ and $B$ be events in a sample space. Then,

$$
Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}
$$

Note:

$$
\frac{Pr[A \cap B]}{Pr[B]} = \frac{\frac{c(A \cap B)}{c(S)}}{\frac{c(B)}{c(S)}} = \frac{c(A \cap B)}{c(B)}
$$
Conditional Probability Formula

Let \( A \) and \( B \) be events in a sample space. Then,

\[
\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}
\]

Note:

\[
\frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\frac{c(A \cap B)}{c(S)}}{\frac{c(B)}{c(S)}} = \frac{c(A \cap B)}{c(B)}
\]
Rolling Two Dice

Example

You roll two dice and note their sum.

1. What is the probability of at least one 3 given that the sum is six?
2. What is the probability that the sum is six given that there is at least one 3?
Example

You roll two dice and note their sum.

1. What is the probability of at least one 3 given that the sum is six?
2. What is the probability that the sum is six given that there is at least one 3?
Rolling Two Dice

Example

You roll two dice and note their sum.

1. What is the probability of at least one 3 given that the sum is six?

\[
\Pr[B|A] = \frac{c(A \cap B)}{c(A)} = \frac{1}{5}
\]

2. What is the probability that the sum is six given that there is at least one 3?
Example

You roll two dice and note their sum.

1. What is the probability of at least one 3 given that the sum is six?

   \[ \Pr[B|A] = \frac{c(A \cap B)}{c(A)} = \frac{1}{5} \]

2. What is the probability that the sum is six given that there is at least one 3?
Rolling Two Dice

Example

You roll two dice and note their sum.

1. What is the probability of at least one 3 given that the sum is six?

   \[
   \Pr[B \mid A] = \frac{c(A \cap B)}{c(A)} = \frac{1}{5}
   \]

2. What is the probability that the sum is six given that there is at least one 3?

   \[
   \Pr[A \mid B] = \frac{c(A \cap B)}{c(B)} = \frac{1}{11}
   \]
On a Used Car Lot

Example

There are 4 vans, 2 SUVs, 6 compacts, and 3 motorcycles on a used car lot. One is chosen at random to be the “special sale” vehicle.

1. What is the probability the van is chosen given that the SUVs are not chosen?
2. What is the probability that the compact is chosen given that only vans or compacts are eligible?
On a Used Car Lot

Example

There are 4 vans, 2 SUVs, 6 compacts, and 3 motorcycles on a used car lot. One is chosen at random to be the “special sale” vehicle.

1. What is the probability the van is chosen given that the SUVs are not chosen?

2. What is the probability that the compact is chosen given that only vans or compacts are eligible?
On a Used Car Lot

Example

There are 4 vans, 2 SUVs, 6 compacts, and 3 motorcycles on a used car lot. One is chosen at random to be the “special sale” vehicle.

1. What is the probability the van is chosen given that the SUVs are not chosen?

\[
\Pr[\text{van} | \text{not SUV}] = \frac{4}{13}
\]

2. What is the probability that the compact is chosen given that only vans or compacts are eligible?
There are 4 vans, 2 SUVs, 6 compacts, and 3 motorcycles on a used car lot. One is chosen at random to be the “special sale” vehicle.

1. What is the probability the van is chosen given that the SUVs are not chosen?
   \[ \Pr[\text{van} \mid \text{not SUV}] = \frac{4}{13} \]

2. What is the probability that the compact is chosen given that only vans or compacts are eligible?
On a Used Car Lot

Example

There are 4 vans, 2 SUVs, 6 compacts, and 3 motorcycles on a used car lot. One is chosen at random to be the “special sale” vehicle.

1. What is the probability the van is chosen given that the SUVs are not chosen?

   \[ \Pr[ \text{van} \mid \text{not SUV} ] = \frac{4}{13} \]

2. What is the probability that the compact is chosen given that only vans or compacts are eligible?

   \[ \Pr[ \text{compact} \mid \text{compact or van} ] = \frac{6}{10} \]
Outline

1. Introduction to Conditional Probability
2. Some Examples
3. A “New” Multiplication Rule
4. Conclusion
Revising the Formula

Revised Conditional Probability Formula

We have seen the formula for conditional probability:

\[ \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} \]

Multiplying both sides by \( \Pr[B] \) yields:

\[ \Pr[A \cap B] = \Pr[B] \cdot \Pr[A|B] \]

Note:

The second formula above allows us to use tree diagrams to compute probabilities using tree diagrams.
Revising the Formula

Revised Conditional Probability Formula

We have seen the formula for conditional probability:

\[
Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}
\]

Multiplying both sides by \(Pr[B]\) yields:

\[
Pr[A \cap B] = Pr[B] \cdot Pr[A|B]
\]

Note:

The second formula above allows us to use tree diagrams to compute probabilities using tree diagrams.
Example

Two urns contain colored balls. The first has 2 white and 3 red balls, and the second has 1 white, 2 red, and 3 yellow balls. One urn is selected at random and then a ball is drawn. Construct a tree diagram showing all probabilities for this experiment.
An experiment consists of 3 steps. First, an unfair coin with $\Pr[H] = \frac{1}{3}$ is flipped. If a heads appears, a ball is drawn from urn #1 which contains 2 white and 3 red balls. If a tails is flipped, a ball is drawn from urn #2 which contains 4 white and 2 red balls. Finally, a ball is drawn from the other urn. Construct a tree diagram to help answer the following questions.

1. What is $\Pr[HWW]$?
2. What is $\Pr[\text{both balls red} | H \text{ flipped}]$?
3. What is $\Pr[\text{1st ball red} | T \text{ flipped}]$?
4. What is $\Pr[\text{last ball red}]$?
Example

An experiment consists of 3 steps. First, an unfair coin with \( \Pr[H] = \frac{1}{3} \) is flipped. If a heads appears, a ball is drawn from urn #1 which contains 2 white and 3 red balls. If a tails is flipped, a ball is drawn from urn #2 which contains 4 white and 2 red balls. Finally, a ball is drawn from the other urn. Construct a tree diagram to help answer the following questions.

1. What is \( \Pr[HWW] \)?
2. What is \( \Pr[ \text{both balls red | H flipped} ] \)?
3. What is \( \Pr[ \text{1st ball red | T flipped} ] \)?
4. What is \( \Pr[ \text{last ball red} ] \)?
Example

An experiment consists of 3 steps. First, an unfair coin with \( \Pr[H] = \frac{1}{3} \) is flipped. If a heads appears, a ball is drawn from urn #1 which contains 2 white and 3 red balls. If a tails is flipped, a ball is drawn from urn #2 which contains 4 white and 2 red balls. Finally, a ball is drawn from the other urn. Construct a tree diagram to help answer the following questions.

1. What is \( \Pr[HWW] \)?
2. What is \( \Pr[ \text{both balls red} \mid H \text{ flipped}] \)?
3. What is \( \Pr[1\text{st ball red} \mid T \text{ flipped}] \)?
4. What is \( \Pr[\text{last ball red}] \)?
An experiment consists of 3 steps. First, an unfair coin with \( \Pr[H] = \frac{1}{3} \) is flipped. If a heads appears, a ball is drawn from urn #1 which contains 2 white and 3 red balls. If a tails is flipped, a ball is drawn from urn #2 which contains 4 white and 2 red balls. Finally, a ball is drawn from the other urn. Construct a tree diagram to help answer the following questions.

1. What is \( \Pr[HWW] \)?
2. What is \( \Pr[ \text{both balls red} \mid H \text{ flipped }] \)?
3. What is \( \Pr[1st \text{ ball red} \mid T \text{ flipped }] \)?
4. What is \( \Pr[\text{last ball red}] \)?
Example

An experiment consists of 3 steps. First, an unfair coin with $\Pr[H] = \frac{1}{3}$ is flipped. If a heads appears, a ball is drawn from urn #1 which contains 2 white and 3 red balls. If a tails is flipped, a ball is drawn from urn #2 which contains 4 white and 2 red balls. Finally, a ball is drawn from the other urn. Construct a tree diagram to help answer the following questions.

1. What is $\Pr[HWW]$?
2. What is $\Pr[\text{both balls red } | \text{ H flipped }]$?
3. What is $\Pr[\text{1st ball red } | \text{ T flipped }]$?
4. What is $\Pr[\text{last ball red }]$?
The next two sections will study questions such as those below in more detail.

**Example**

In the previous example, find \( Pr[ \text{last ball red} \mid \text{H flipped} ] \) and \( Pr[ \text{last ball red} \mid \text{T flipped} ] \). Does the result of the coin toss change the probability that the last ball is red?

**Example**

Again using the previous example, find \( Pr[ \text{H flipped} \mid \text{last ball red} ] \). Can the tree diagram be used to find this probability?
Preparing for Next Time

The next two sections will study questions such as those below in more detail.

Example
In the previous example, find \( \Pr[ \text{last ball red} \mid \text{H flipped} ] \) and \( \Pr[ \text{last ball red} \mid \text{T flipped} ] \). Does the result of the coin toss change the probability that the last ball is red?

Example
Again using the previous example, find \( \Pr[ \text{H flipped} \mid \text{last ball red} ] \). Can the tree diagram be used to find this probability?
Preparing for Next Time

The next two sections will study questions such as those below in more detail.

**Example**

In the previous example, find $\Pr[\text{last ball red | H flipped}]$ and $\Pr[\text{last ball red | T flipped}]$. Does the result of the coin toss change the probability that the last ball is red?

**Example**

Again using the previous example, find $\Pr[\text{H flipped | last ball red}]$. Can the tree diagram be used to find this probability?
Introduction to Conditional Probability

Some Examples

A "New" Multiplication Rule

Conclusion
Conditional Probability Formula:

\[ \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} \]

Using tree diagrams for probability:

\[ \Pr[A \cap B] = \Pr[B] \cdot \Pr[A|B] \]
Important Concepts

Things to Remember from Section 7-4

1. Conditional Probability Formula:

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$$

2. Using tree diagrams for probability:

$$Pr[A \cap B] = Pr[B] \cdot Pr[A|B]$$
Conditional Probability Formula:

\[ \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} \]

Using tree diagrams for probability:

\[ \Pr[A \cap B] = \Pr[B] \cdot \Pr[A|B] \]
Next time we will introduce the concept of “independent events” and how they relate to conditional probabilities.

For next time

- Read Section 7-5
- Prepare for Quiz on 7-4
Next time we will introduce the concept of “independent events” and how they relate to conditional probabilities.

For next time
- Read Section 7-5
- Prepare for Quiz on 7-4