MATH 112
Section 3.3: Understanding Multiplication

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Learning about Multiplication

What do you remember learning about multiplication in elementary school? There are two aspects of multiplication which are important.

- The structure and algorithms for multiplication
- Memorization of multiplication tables

Example

There are several types of multiplication problems.

- One soda costs $0.75. How much does 3 sodas cost?
- If Tom drives 20 mph for 3 hours, how far does he travel?
- A rug is 3 feet by 4 feet. How much area is the rug?
- A pizza comes with one of 3 sauces and one of 5 toppings. How many pizzas are possible?
Repeated Addition Model

One of the most basic models of multiplication is the repeated addition model.

Example

The vending machine charges $0.75 for a can of sodas. How much will you need to purchase a total of 3 sodas from the machine?

To find the solution, add $0.75 three times:

\[ 3 \times 0.75 = 0.75 + 0.75 + 0.75 = 2.25 \]

Repeated Addition

In general, to multiply \( n \times a \), add together \( n \) factors of \( a \).

\[ n \times a = a + a + \cdots + a \]

\[ n \text{ times} \]
Distance Model

As was the case with addition, we can the concept of distance on a number line to model multiplication.

Example

Tom, wanting to avoid his 5th speeding ticket in as many months, drives to work traveling 20 miles per hour. If it takes him 3 hours to get to work, how far away is his workplace?

This model is similar to the repeated addition model, but is shown visually on a number line. If Tom travels 20 miles per hour for three total hours, he travels 60 miles.

In general, to multiply $n \times a$, place $n$ $a$'s side by side and measure the total distance.
Area Model

Multiplication can also be modeled as area. We used this concept when we looked at multiplying polynomials using algebra tiles.

Example

You wish to make a rug for your front entry way. The rug needs to be three feet wide by four feet long. How much material will you need to purchase for the rug?

Draw a picture of the rug which is three feet wide by four feet long. Count the number of square feet in the rug to get the total of 12 square feet.

Area Model

In general, to multiply $n \times a$, draw a rectangle which is $n$ blocks wide by $a$ blocks long and count the number of blocks in the rectangle.
Tree Model

The final model which we will examine today is called the “tree” model or the “Cartesian Product” model.

Example

A pizza comes with your choice of one of three sauces and one of five toppings. How many different pizzas are possible using this menu?

Draw a tree with one branching for the sauce, and from each end another branching for the toppings. The total number of branches at the end is $3 \times 5 = 15$.

Tree/Set Model

In general, if $A$ is a set of $a$ elements and $B$ is a set of $b$ elements, then $a \times b$ is the number of elements in

$$\{ (x, y) \mid x \in A \text{ and } y \in B \}.$$
Properties of Multiplication

Multiplication, like addition, has several properties which are of interest.

Properties of Multiplication

For whole numbers $a$, $b$, and $c$, the following properties hold.

- **Identity Property**: $1 \times a = a \times 1 = a$
- **Commutative Property**: $a \times b = b \times a$
- **Associative Property**: $(a \times b) \times c = a \times (b \times c)$
- **Zero Property**: $a \times 0 = 0 \times a = 0$
- **Distributive Property**: $a \times (b + c) = a \times b + a \times c$
One difference between addition or subtraction and multiplication is what happens to the units of each number involved.

**Example**

As in one of our opening examples, 20 miles per hour times 3 hours equals 60 miles.

**Example**

In one of our other examples, a rug that is 3 linear feet wide and 4 linear feet long has a total of 12 square feet of area.

**Example**

On a certain block there are 2 cats per house. If there are 5 houses on this block, then there are a total of 10 cats.
Modeling Base 10 Multiplication

Multiplication can be modeled with base 10 blocks using an area model similar to our rug example. This is accomplished in much the same way as was done with the algebra tiles we saw in lab.

Example

Use base 10 blocks to model each multiplication problem.

\[
\begin{array}{c}
24 \\
\times 3 \\
\end{array}
\begin{array}{c}
31 \\
\times 25 \\
\end{array}
\]

\[
\begin{array}{c}
72 \\
\end{array}
\begin{array}{c}
775 \\
\end{array}
\]
The Standard Algorithm

As with addition and subtraction, you all probably know the algorithm I will refer to as the “standard” multiplication algorithm.

Standard Multiplication Algorithm

In the standard algorithm, multiplication is done starting with the digit on the bottom right. This digit is multiplied by each digit in the top row from right to left. Products with more than one digit are carried over to the next column. Each new digit on the bottom row produces another product which is written one place to the left. These products are then added to get the final product.

Example

Use the standard multiplication algorithm to find the following product.

\[
\begin{array}{c}
3 & 2 & 1 \\
\times & 4 & 3 \\
\end{array}
\]
Lattice multiplication is related to lattice addition. However, the lattice is an even more useful tool in multiplication problems.

To multiply using lattice addition, create a box with one column for each digit in the first number and one row for each digit in the second number. After creating a lattice in this box, record the product of the digit for each column/row combination in the corresponding box. Finally, add down the diagonals to get the final product.

Example

Use lattice multiplication to find the following product.

\[
\begin{array}{c}
3 & 2 & 1 \\
\times & 4 & 3 \\
\end{array}
\]
One of the most unusual alternative algorithms we will see in this class is called the Russian Peasant’s algorithm for multiplication.

**Russian Peasant’s Algorithm**

To multiply using the Russian Peasant’s algorithm, start with both numbers at the top of a two-column table. Then, start halving the larger number while you double the smaller number. If the number to be halved is odd, round down. Finally, add the numbers from the doubled column which correspond to odd numbers in the halved column. This is the product.

**Example**

Use the Russian Peasant’s algorithm to find the following product.

\[
\begin{array}{c c c}
3 & 2 & 1 \\
\times & 1 & 5 \\
\end{array}
\]
As with addition and subtractions, using some clever manipulations can make multiplications easier to do mentally.

**Example**

Find each product mentally as quickly as possible.

1. \(37 \times 5\)
2. \(15 \times 12\)
Important Concepts

Things to Remember from Section 3.3

1. Interpretations of Multiplication
2. Properties of Multiplication
3. Alternative Multiplication Algorithms
4. Mental Multiplication Techniques