MATH 113
Section 5.4: Decimals, Exponents, and Real Numbers

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Outline

1 Introduction
2 Modeling with Decimals
3 Irrational Numbers
4 Conclusion
Decimals

Decimals are an even more recent invention than fractions.

History of Decimals

In 1585 Flemish mathematician Simon Stevin published a small pamphlet, La Thiende (“The Tenth”), in which he presented an account of decimal fractions and their daily use. Though he did not invent decimal fractions and his notation was clumsy, he established the use of decimals in day-to-day mathematics.

Decimal Notation

To this day decimal notation varies.

- United States: 3.14
- United Kingdom: 3·14
- Continental Europe: 3,14
Decimals and Place Value

In order to tie decimals into our existing number system, we will start with an examination of place values.

Decimal Place Values

In base 10, decimals have place values which are negative powers of 10, or positive powers of $\frac{1}{10}$.

\[
10^2 \quad 10^1 \quad 10^0 \quad \ldots \quad 10^{-1} \quad 10^{-2}
\]

Example

Suppose we used a base 5 system. What would the “decimal” place values look like?
Decimals and Fractions

As decimals were originally intended to simplify working with fractions, there is a strong connection between the two concepts.

Decimals are an alternative to fractions with denominators which are powers of 10. For example, \(0.3 = 0 \times 1 + 3 \times \frac{1}{10} = \frac{3}{10}\).

Example

Convert each fraction to a decimal:

\[
\begin{array}{cccc}
\frac{1}{2} & \frac{1}{4} & \frac{1}{16} & \frac{1}{3} \\
\end{array}
\]

Example

Convert each decimal to a fraction:

\[
\begin{array}{ccc}
0.3 & 0.045 & 0.23 \\
\end{array}
\]
The Importance of 0

Zero plays an important role in our numbering system. This role is especially important when writing decimal numbers.

The Role of 0

Zero (0) acts as a place holder in our numeration system. In decimal numbers, leading zeros can be especially important.

Cases where Zeros Make A Difference

Note the following:

\[ 203 \neq 23 \quad .203 \neq .23 \quad .023 \neq .23 \]

Cases where Zeros are Optional

Note the following:

\[ 0.23 = .23 \quad .230 = .23 \]
As we saw in the lab, base 10 blocks can be used to help model decimals as long as we are sure to designate the unit.

**Example**

Model the following decimals using base 10 blocks.

1. 0.29
2. 0.032
3. 0.30

Sometimes models, converting to fractions, or padding with zeros can help us compare decimals.

**Example**

Arrange the three decimals from the previous example in descending order.
Modeling Decimal Addition

Modeling decimal addition is relatively straightforward. Once can simply use base 10 blocks and the set model of addition.

**Example**

Use base 10 blocks to model and find $6.5 + 1.25$.

Models can also help to justify certain important practices in our standard addition algorithm.

**Example**

Using blocks, explain why we need to line up the decimal point before we add two decimal numbers together.
Modeling Decimal Multiplication

The area model works well for showing decimal multiplication. Again, we must be sure to designate a unit and we must also differentiate between linear units and square units.

Example

Use an area model to find $3.4 \times 2.3$.

Example

Now perform the multiplication $3.4 \times 2.3$ using the standard multiplication algorithm. Can you justify this algorithm using blocks? In particular, why do we count up the number of decimal places in the factors to place the decimal in the product?
Modeling Decimal Division

Just as with fractions, the repeated subtraction model can help us see how to divide decimals.

**Example**

Use base 10 blocks and repeated subtraction to find $4.5 \div 2.1$. Be careful in expressing any remainder.

**Example**

Now perform the division $4.5 \div 2.1$ using the standard division algorithm. Can you justify this algorithm using blocks? In particular, why can we move the decimals in the divisor and dividend?
A New Operation – Exponentiation

Although we have been using the concept of an exponent to find place values, we have yet to formally introduce this operation.

### Multiplication as Repeated Addition

Recall that the product $a \times b$ can be thought of as:

$$a \times b = b + b + \cdots + b$$

$a$ times

### Exponentiation

For any number $b$ and an integer $a$, the expression $b^a$ means that we multiply $a$ factors of $b$ together. $a$ is called the exponent and $b$ is called the base in this expression. Symbolically:

$$b^a = b \times b \times \cdots \times b$$

$a$ times
Exponentiation and Scientific Notation

**Example**

Find the value of $2^a$ for $a = 4, 3, 2, 1, 0, -1, \text{ and } -2$.

Exponentiation is used together with decimals to write numbers in a form called Scientific Notation.

**Scientific Notation**

In scientific notation numbers are written as $d \times 10^a$ where $d$ is a decimal number and $a$ is any integer.

**Example**

Convert between scientific and standard notation as appropriate.

1. $3,425,471$
2. $0.00000000146$
3. $1.25 \times 10^6$
4. $2.13 \times 10^{-7}$
Are Decimals and Fractions Different?

As we saw at the beginning of the class, decimals were intended to make working with fractions easier. However, are decimals and fractions really the same thing?

**Example**

Can every fraction be written as a decimal? If so, what types of decimals? If not, why not?

**Example**

Can every decimal be written as a fraction? If so, justify your answer. If not, give an example.
Proof that $\sqrt{2}$ is Irrational

Numbers which can not be written as fractions include $\pi$ and $\sqrt{2}$.

Example

Show that $\sqrt{2}$ is not a rational number.

- If it were, then there are integers $a, b$ with $\sqrt{2} = \frac{a}{b}$ where $a$ and $b$ have no common factors.
- Multiplying by $b$, $\sqrt{2}b = a$.
- Squaring, $2b^2 = a^2$.
- We see from the above that $a^2$ is even. Therefore, $a$ is also even.
- If $a$ is even, then $a^2$ is a multiple of 4, say $a^2 = 4k$.
- Simplifying, $b^2 = 2k$
- As before, $b^2$ must be even, so $b$ is also.
- But then $a$ and $b$ are both even, so they have a common factor of 2.
- Since this can’t happen, there are no $a, b$ with $\sqrt{2} = \frac{a}{b}$.
Irrational Numbers

Now that we have shown that there are numbers which can not be written as fractions, we are ready for our final Venn Diagram.

Number Sets

- $\mathbb{R}$ - Real Numbers (all numbers studied in this class)
- $\mathbb{I}$ - Irrational Numbers (can not be written as whole fractions)
- $\mathbb{Q}$ - Rational Numbers (can be written as whole fractions)
- $\mathbb{Z}$ - Integers (+/− whole numbers)
- $\mathbb{W}$ - Whole Numbers (0, 1, 2, 3, . . .)
- $\mathbb{N}$ - Natural Numbers (1, 2, 3, . . .)
Important Concepts

Things to Remember from Section 5.4

1. Decimal notation and place values
2. Conversion between decimals and fractions
3. Modeling operations with decimals
4. Exponents and scientific notation
5. Irrational numbers (definitions and examples)