MATH 113
Section 5.2: Fractions and Rational Numbers

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Winter Quarter, 2008
Now that we have discussed the set of integers, it is time to expand our set of numbers yet again with the addition of rational numbers.

**Rational Numbers**

The set of rational numbers is the set of all ratios of two integers where the second integer is not zero. That is:

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

Although these are commonly referred to as “fractions” we need to be careful. A fraction is a ratio of any two numbers. For example, \(\frac{\sqrt{2}}{2}\) is a fraction but not a rational number. Rational numbers are ratios of integers only.
The rational numbers contain the integers (for example, $\frac{3}{1}$ is 3). We can represent this using a Venn diagram as follows.

Note that in our definition of rational numbers, there are many different ways to represent each number. Find two more ways to represent the integer 3 as a rational number.

Rational numbers are the first set of numbers we have seen which are dense. That is, between any two rational numbers, there is another.
A Brief History of Rational Numbers

Because rational numbers are a more complicated concept than whole numbers, they are a relatively recent innovation.

Rational Number Facts

- Egyptians used fractions, but only “unit fractions” of the form $\frac{1}{n}$. These were denoted by placing an oval over the symbolic representation of the denominator.

- Greeks did not like non-whole numbers so they used ratios instead of fractions. For example, to denote the idea that $\frac{2}{5}$ths of a pie has been eaten, Greeks would say the ratio of the pie that was eaten to the pie that remained is 2 : 3.
Three Fraction Models

There are many different ways to model fractions. We will focus on three general types of models.

Example

Draw a picture to model the fraction \( \frac{4}{5} \).

In the **area model** a figure is divided into 5 pieces and 4 of those are shaded.

In the **linear model** a number line is divided into 5 pieces and the first 4 are covered.

In the **set model** 5 objects are drawn and 4 of them are circled.
Selecting an Appropriate Model

Fractions are used in many different situations and some models are more appropriate than others depending on the situation.

Example

Determine which model is most useful for each situations.

- **Fractions and Measure**
  
  I went \(\frac{4}{5}\)th of the total distance.

- **Fractions as Quotients**
  
  I have 4 cookies which I want to split between 5 people.

- **Fractions as an Operation**
  
  \(\frac{4}{5}\)ths of your class of 20 people missed Friday’s class.

- **Fractions as Ratios**
  
  \(\frac{4}{5}\)ths of your ideas are good ones.

What properties or ideas are common to each situation?
Fractions and Units

In many situations we model fractions by representing the unit (number 1) as a whole object and then dividing it into equal pieces. However, there are situations when it is more convenient to use some other number as the whole and divide it into pieces.

Example

Use the set model below to represent $\frac{2}{3}$rd. Justify this in two different ways.

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet = 2
\end{array}
\]
Using Fraction Models

Fraction models are useful for more than just representing a fraction using pictures. They can also be used to help formulate the solution to a story problem.

Example

You paid $6 for $1\frac{1}{2}$ pound of potato salad. What is the cost of one pound of potato salad?

While there are several ways to approach this problem, using an area model to represent a whole of $1\frac{1}{2}$ pounds we can find the solution.

If you do not wish to draw a picture, you can solve this problem using ratios.
More Story Problems

Example
Sally has some free time to split between exercising and reading. She spends $\frac{1}{6}$ of her time exercising. If she spends 2 hours more reading than she did exercising, how much free time did she have?

Example
A thief stole some apples from an orchard. On his way out, he met 3 watchmen, one after the other. He gave each watchman $\frac{1}{2}$ of the apples that he had at the time and two more besides. He escaped with one apple. How many apples did he originally steal?

Example
A business man spends $\frac{1}{3}$ of his travel budget on food. He splits this evenly between groceries and dining out. If he spends $20 on each of the 6 meals he eats out, what is his travel budget?
Defining Equivalence

At the beginning of the class, we commented that there are many different ways to represent a single number using fractions.

Equivalent Fractions

Fractions are called equivalent if they represent the same value.

Example

In each example, use two different models to show that the fractions are equivalent.

1. \( \frac{3}{4} = \frac{9}{12} \)
2. \( \frac{3}{2} = \frac{15}{10} \)

Procedure

What is the procedure which we follow to determine if \( \frac{3}{4} \) and \( \frac{9}{12} \) are equivalent?
Simplest Form

Since there are so many different ways to write a single rational number, it is convenient to have a single preferred form.

**Simplest Form**

A fraction is in simplest form (also called reduced) if the numerator and denominator are relatively prime.

**Reducing to Simplest Form**

There are several methods to reduce fractions to simplest form.

- Guess and Check
- Recognizing Factors
- Prime Factorization

**Example**

Reduce \(\frac{56}{32}\) to simplest form.
Important Concepts

Things to Remember from Section 5.2

1. The definition of rational numbers and how they relate to other number sets

2. Three models for fractions:
   - Area model
   - Linear model
   - Set model

3. Using fraction models to solve story problems

4. Identifying equivalent fractions