Exam I - Answers to Review Sheet
MATH 250, Spring 2003

1. Review the proofs from your homework assignment, especially the adaptation of the proof that \( \sqrt{2} \) is irrational to the \( \sqrt{n} \) case.

2. Provide a simple rule for each sequence, and find the value of each sum.
   (a) \[ \frac{1 + (-1)^{n-1}}{2} \cdot 2^{\frac{n-1}{2}} \]
   (b) \( n! + 1 \)
   (c) 10
   (d) 110

3. Prove the following using mathematical induction
   (a) Base Case: For \( n = 1 \), \( \sum_{j=1}^{n} j^4 = 1 = 1(1 + 1)(2 + 1)(3 + 3 - 1)/30 \)
   Inductive Step: Assume now that the formula holds for \( n \). We show that \( \sum_{j=1}^{n+1} j^4 = (n+1)(n + 2)(2n + 3)(3(n + 1)^2 + 3(n + 1) - 1)/30 \)
   (b) Done in class
   (c) Base Case: Clearly 5 divides \( (0^5 - 0) = 0 \).
   Inductive Step: Now, suppose that 5 divides \( n^5 - n \). We show that 5 must divide \( (n+1)^5 - (n+1) \)
   Now \( (n+1)^5 - (n+1) = (n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) - (n+1) = (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n) \).
   So that since 5 divides \( n^5 - n \), and 5 clearly divides the second term, 5 divides the entire sum, as desired.

4. Perform the following tasks dealing with recursive functions.
   (a) \( a_n = a_{n-1} + 6 \) with \( a_0 = 0 \)
   \( a_n = 10(a_{n-1}) \) with \( a_0 = 1 \)
   \( a_n = a_{n-1} + 2 \) with \( a_0 = 1 \)
   (b) Base: \( 2 \in S \), Recursive Step: If \( x \in S \) then \( x + 3 \in S \).
   (c) \( S \) is the set of all binary strings.

5. Use basic counting rules to determine the number of ways the following situations can occur.
   (a) \( 6 \times P(9, 5) = 6 \times 15 = 90, 720 \)
   \( C(6, 2) \times P(8, 4) = 15 \times 1, 680 = 25, 000 \)
   \( 6 \times P(8, 5) + 6 \times P(8, 5) = 2 \times 6 \times 1, 680 = 20, 160 \)
   (b) \( 2^5 + 2^4 - 2^2 = 32 + 16 - 4 = 44 \)
   (c) If \( n \) is even, then \( 2^\frac{n+1}{2} \), otherwise \( 2^{\frac{n+1}{2}} \)

6. Use the pigeonhole principle to determine the following.
   (a) 9 cards
   (b) 4 numbers
   (c) 101
7. Use permutations or combinations to count the following.
   (a) $C(52, 5) = 2,598,960$
   (b) $P(25, 10) = 11,861,876,288,000$
   (c) $C(6, 2) \times C(6, 3) = 15 \times 20 = 300$
   (d) $P(12, 2) \times C(10, 3) = 12 \times 10 \times 120 = 14,400$

8. Find a recurrence relation and initial conditions for each of the following situations. Then solve those recurrence relations.
   (a) $a_n = n(a_{n-1})$ with $a_0 = 1$
   (b) $a_n = 4a_{n-1}$ with $a_1 = 10$
   (c) $a_n = a_{n-1} + 2a_{n-2}$ with $a_1 = 3$ and $a_2 = 7$

9. Use generating functions to answer the following.
   (a) $\frac{1}{(1+x)^7} = \sum_{k=0}^{\infty} kx^k$, so coefficient of $x^{10}$ is 10.
   (b) coefficient of $x^{12}$ in $(1 + x + x^2 + x^3)^5$
   (c) $a_kx^k = 5a_{k-1}x^k - 6a_{k-2}x^k$
      $\sum_{k=3}^{\infty} a_kx^k = 5\sum_{k=2}^{\infty} a_{k-1}x^k - 6\sum_{k=2}^{\infty} a_{k-2}x^k$
      $G(x) - 6 - 30x = 5x\sum_{k=1}^{\infty} a_kx^k - 6x^2\sum_{k=0}^{\infty} a_kx^k$
      $G(x) - 6 - 30x = 5x(G(x) - 6) - 6x^2G(x)$
      $6x^2G(x) - 5xG(x) + G(x) = 6$
      $G(x)(6x^2 - 5x + 1) = 6$
      $G(x) = \frac{6}{(3x-1)(2x-1)} = \frac{6}{(1-3x)(1-2x)}$
      $G(x) = \frac{18}{1-3x} - \frac{12}{1-2x}$
      $G(x) = 18\left(\sum_{k=0}^{\infty} 3^kx^k\right) - 12\left(\sum_{k=0}^{\infty} 2^kx^k\right)$
      $G(x) = \sum_{k=0}^{\infty} (18(3^k) - 12(2^k))x^k$
      so, $a_k = 18(3^k) - 12(2^k)$