Exam II - Answers to Review Sheet
MATH 250, Spring 2004

1. Provide a simple rule for each sequence, and find the value of each sum.
   (a) \( \left( \frac{1+(-1)^{n-1}}{2} \right) 2^{\frac{n-1}{2}} \)
   (b) \( n! + 1 \)
   (c) 10
   (d) 126

2. Prove the following using mathematical induction
   (a) Base Case: For \( n = 1 \), \( \sum_{j=1}^{n} j^4 = 1 = (1 + 1)(2 + 1)(3 + 1)/30 \)
   Inductive Step: Assume now that the formula holds for \( n \). We show that \( \sum_{j=1}^{n+1} j^4 = (n + 1)(n+2)(2n+3)(3n^2 + 3n - 1)/30 \)
   (b) Done in class
   (c) Base Case: Clearly 5 divides \( (0^5 - 0) = 0 \).
   Inductive Step: Now, suppose that 5 divides \( n^5 - n \). We show that 5 must divide \( (n+1)^5 - (n+1) \)
   Now \( (n+1)^5 - (n+1) = (n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) - (n+1) = (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n) \).
   So that since 5 divides \( n^5 - n \), and 5 clearly divides the second term, 5 divides the entire sum, as desired.

3. Perform the following tasks dealing with recursive functions.
   (a) \( a_n = a_{n-1} + 6 \) with \( a_0 = 0 \)
   \( a_n = 10(a_{n-1}) \) with \( a_0 = 1 \)
   \( a_n = a_{n-1} + 2 \) with \( a_0 = 1 \)
   (b) Base: \( 2 \in S \). Recursive Step: If \( x \in S \) then \( x + 3 \in S \).
   (c) \( S \) is the set of all binary strings.

4. Use basic counting rules to determine the number of ways the following situations can occur.
   (a) \( 6 \times P(9, 5) = 6 \times 15, 120 = 90, 720 \)
   \( C(6, 2) \times P(8, 4) = 15 \times 1, 680 = 25, 200 \)
   \( 6 \times P(8, 5) + 6 \times P(8, 5) = 2 \times 6 \times 1, 680 = 20, 160 \)
   (b) \( 2^5 + 2^4 - 2^2 = 32 + 16 - 4 = 44 \)
   (c) If \( n \) is even, then \( 2^{\frac{n+1}{2}} \), otherwise \( 2^{\frac{n+1}{2}} \)

5. Use the pigeonhole principle to determine the following.
   (a) 9 cards
   (b) 4 numbers
   (c) 101

6. Use permutations or combinations to count the following.
   (a) \( C(52, 5) = 2, 598, 960 \)
   (b) \( P(25, 10) = 11, 861, 876, 288, 000 \)
   (c) \( C(6, 2) \times C(6, 3) = 15 \times 20 = 300 \)
   (d) \( P(12, 2) \times C(10, 3) = 12 \times 11 \times 120 = 15, 840 \)
7. Find a recurrence relation and initial conditions for each of the following situations. Then solve those recurrence relations.

(a) \( a_n = n(a_{n-1}) \) with \( a_0 = 1 \)
(b) \( a_n = 4a_{n-1} \) with \( a_1 = 10 \)
(c) \( a_n = a_{n-1} + 2a_{n-2} \) with \( a_1 = 3 \) and \( a_2 = 7 \)

8. Use generating functions to answer the following.

(a) \( \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}kx^k \), so coefficient of \( x^{10} \) is 10.
(b) coefficient of \( x^{12} \) in \( (1 + x + x^2 + x^3)^5 \)
(c) \[
\begin{align*}
\sum_{k=0}^{\infty} a_k x^k &= 5\sum_{k=2}^{\infty} a_{k-1} x^k - 6\sum_{k=2}^{\infty} a_{k-2} x^k \\
G(x) - 6 - 30x &= 5x\sum_{k=1}^{\infty} a_k x^k - 6x^2\sum_{k=0}^{\infty} a_k x^k \\
6x^2G(x) - 5xG(x) + G(x) &= 6 \\
G(x)(6x^2 - 5x + 1) &= 6 \\
G(x) &= \frac{6}{(3x - 1)(2x - 1)} \\
G(x) &= \frac{6}{(1 - 3x)(1 - 2x)} \\
G(x) &= \frac{18}{1 - 3x} - \frac{12}{1 - 2x} \\
G(x) &= 18\left(\sum_{k=0}^{\infty} 3^k x^k\right) - 12\left(\sum_{k=0}^{\infty} 2^k x^k\right) \\
G(x) &= \sum_{k=0}^{\infty} \left(18(3^k) - 12(2^k)\right)x^k \\
a_k &= 18(3^k) - 12(2^k)
\end{align*}
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