Exam III - Answers to Review Sheet  
MATH 250, Spring 2004

1. Determine which of the properties: Reflexive, Symmetric, Antisymmetric, and Transitive, is enjoyed by each of the following relations.

(a) Not reflexive, symmetric, antisymmetric, or transitive.
(b) Not reflexive or transitive but symmetric.
(c) Not reflexive, symmetric or transitive, but antisymmetric.
(d) Reflexive, symmetric and transitive (not antisymmetric of course).

2. Using the relations above, find each of the following new relations.

(a) All pairs \((n, m)\) in which \(n = m^2\) or \(n\) and \(m\) have the same remainder when divided by 3.
(b) All pairs \((n, k)\) where \(k\) is congruent to the square root of \(m\) mod 3.
(c) All pairs \((a, b)\) where \(a\) is a teacher of student \(b\).
(d) \(\{(\square, \square), (\square, \nabla), (\heartsuit, \heartsuit), (\heartsuit, \nabla), (\nabla, \nabla), (\nabla, \square), (\nabla, \heartsuit), (\nabla, \nabla)\}\)

3. Suppose that the following relations are given. Give the sequence of relational operations on the relation required to produce each new relation described below.

\[
R = \begin{array}{cccc}
\text{student_id} & \text{name} & \text{major} & \text{class} \\
10 & david & math & freshman \\
15 & sarah & comp. sci. & senior \\
14 & david & engineering & junior \\
\end{array}
\]

\[
S = \begin{array}{cc}
\text{club_name} & \text{student_id} \\
\text{math club} & 10 \\
\text{math club} & 15 \\
\text{cs club} & 15 \\
\end{array}
\]

(a) \(\Pi_{\text{name}}(R)\)
(b) \(\Pi_{\text{name}}(\sigma_{\text{major} = \text{"math"}}(R))\)
(c) \(\Pi_{\text{name}}(\sigma_{\text{class} = \text{"senior"}} (\sigma_{\text{club_name} = \text{"cs club"}} (\sigma_{\text{R.student_id} = \text{S.student_id}} (R \times S))))\))
(d) \(\Pi_{\text{name}}(\sigma_{\text{club_name} = \text{"cs club"}} (\sigma_{\text{C}}(R \times S)))) \cup \Pi_{\text{name}}(\sigma_{\text{club_name} = \text{"math"}} (\sigma_{\text{C}}(R \times S))))\)
where \(C = R_{\text{student_id}} = S_{\text{student_id}}.\)

4. Represent each of the following relations as a matrix and a digraph. Use the matrix representation to determine if the relation is reflexive and/or symmetric. Use the digraph representation to check transitivity. Assume that the relations are all on the set \(\{a, b, c, d, e\}\)

(a) \[
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Not reflexive, symmetric, or transitive.

(b) \[
\begin{bmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
Not reflexive or transitive, but symmetric.
\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
(c) Not reflexive but antisymmetric and transitive.

5. Left to the reader

6. Consider the relation \( R = \{ (x, y) \mid x - y \text{ is an integer} \} \).
   (a) \( R \) is reflexive since \( x - x = 0 \) is an integer, symmetric since \( x - y = -(y - x) \), and transitive since if \( x - y \) is an integer and \( y - z \) is an integer then so is \( x - z \).
   (b) \([1]_R = \mathbb{Z}\) and \([\frac{1}{2}]_R = \{ \frac{n}{2} \mid n \text{ is an odd integer} \}\)

7. Answers may vary.

8. See your text for these graphs.

9. For each of the graphs above, determine:
   (a) one: directed pseudo multi-graph, two: simple graph, three: simple graph.
   (b) left to the student
   (c) left to the student (see matrices and graphs in relation example above).
   (d) one: not weakly connected, but strongly connected.

10. Moving from right to left, top to bottom the (in,out) degrees are: (3,2), (1,1), (1,2), (1,2), (2,0)

11. No: graph three has a vertex of degree 4, graph two does not.

12. The adjacency and path matrices are:

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 5 & 5 & 0 & 6 & 6 \\
5 & 2 & 3 & 2 & 1 & 1 \\
5 & 3 & 2 & 2 & 1 & 1 \\
0 & 2 & 2 & 0 & 4 & 4 \\
6 & 1 & 1 & 4 & 0 & 0 \\
6 & 1 & 1 & 4 & 0 & 0 \\
\end{bmatrix}
\]