Final Exam Review Sheet
MATH 250, Spring 2004

This exam will cover sections 1.1-1.6, 2.1-2.4, 4.1-4.3, 5.1-5.3, 7.1, 7.2, 7.4, 8.1-8.5, 9.1-9.5, 10.1, 10.3, and 10.4 in your text. You should know general terms, definitions and theorems from each of these sections. In preparation for the exam, you should review the homework and previous exams, and prepare a one page, 8.5 × 11, handwritten page with definitions, theorems, and formulas (but no worked examples) which you may use during the exam.

1. Chapters 1 and 2
   (a) Translation between formal logic (predicate or propositional) and English.
   (b) Showing logical equivalences with truth tables or identities.
   (c) Symbolic proofs using the rules of inference in either two-column or natural deduction form.
   (d) Proving basic theorems of algebra via direct, indirect, contradiction, equivalence, or proof-by-cases methods.
   (e) Working with set unions, intersections, cross products, power sets, symmetric differences, etc.
   (f) Showing sets $A$ and $B$ are equal using either logical equivalence or showing $A \subseteq B$ and $B \subseteq A$.
   (g) Functions, injections, surjections, bijections, function composition, etc.
   (h) Using functions to define and compare the cardinality of sets.

2. Chapter 4
   (a) Proofs by mathematical induction, strong induction, and the well ordering principle.
   (b) Recursively defined sets and functions and structural induction.

3. Chapters 5 and 7
   (a) The sum and product counting rules.
   (b) The pigeonhole principle.
   (c) Counting with permutations and combinations.
   (d) Defining recurrence relations for a specific counting problem.
   (e) Solving homogeneous recurrence relations with characteristic equations.
   (f) Solving non-homogeneous recurrence relations using particular solutions.
   (g) Solving recurrence relations using generating functions.

4. Chapter 8
   (a) Working with relations, relation properties, representations, and closures.
   (b) $n$-ary relations and relational algebra.
   (c) Equivalence relations, equivalence classes, and partitions.

5. Chapters 9 and 10
   (a) Classification of graphs and representation using an adjacency or incidence matrix.
   (b) Graph isomorphisms.
   (c) Euler and Hamiltonian circuits and paths.
   (d) Definition and properties of trees.
   (e) Using trees to represent post/in/pre-fix arithmetic or logical sentences, and tree traversals.
Below is a list of *sample* problems. This list is not all-inclusive, but does represent the types of problems you will see on the exam.

1. Let \( t \): I train for a marathon, \( R(x) \): I run in marathon \( x \), and \( W(x) \): I win marathon \( x \). Translate between English and formal logic as appropriate.
   (a) I train for a marathon, but do not run in any.
   (b) If I train for a marathon and run in the College Place marathon, then I will win it.
   (c) \( t \rightarrow \exists x R(x) \)
   (d) \( \forall x (R(x) \land \neg W(x)) \leftrightarrow \neg t \)

2. Determine if the pairs of compound propositions are logically equivalent with and without truth tables.
   (a) \( (p \rightarrow q) \lor (p \rightarrow r) \) and \( p \rightarrow (q \lor r) \)
   (b) \( p \rightarrow \neg(q \rightarrow r) \) and \( \neg((\neg p \lor q) \rightarrow (p \land r)) \)
   (c) \( (q \land (\neg p \rightarrow q)) \rightarrow p \) and \( F \) (i.e. the first expression is a contradiction).

3. Determine if the following pairs of quantified predicate statements are logically equivalent, where \( A \) is a proposition not involving quantifiers.
   (a) \( (\forall x P(x)) \land A \) and \( \forall x (P(x) \land A) \)
   (b) \( \forall x (P(x) \rightarrow Q(x)) \) and \( \exists x (P(x) \rightarrow Q(x)) \)
   (c) \( \forall x (P(x) \lor Q(x)) \) and \( \forall x P(x) \lor \forall x Q(x) \)

4. Perform the following proofs symbolically, if they are valid. If they are not valid, state the reason.
   (a) “Randy plays on Tuesdays,” “If Randy plays then Randy does not work,” and “If Randy does not work, Randy does not get paid.” imply “Randy does not get paid on Tuesdays.”
   (b) “Every C.S. major wants to take Discrete Math,” “All C.S. majors want to be Math majors,” and “Top wants to take Discrete Math,” therefore “Tom wants to be a Math major.”

5. Answer each of the following questions related to the sets \( A = \{x, y\} \) and \( B = \{1, 2, 3\} \).
   (a) What are the elements of \( \mathcal{P}(A) \)?
   (b) What are the elements of \( \mathcal{P}(A \times B) \)?
   (c) What are the sets \( (A \setminus B \cup B \setminus A), (A \setminus B \cap B \setminus A) \), and \( A \oplus B \)?

6. Suppose that \( f \) and \( g \) are functions from \( A \) to \( B \) above. Can \( f \) be an injection, surjection, or bijection? Find a function \( f \) and a distinct function \( g \), both injective, and then find \( f \circ g \). Is it also injective?

7. Prove by induction that if \( a \equiv 0 \pmod{5} \), then \( a^n \equiv 0 \pmod{5} \).

8. Perform the following tasks dealing with recursive functions.
   (a) Give recursive definitions of the sequences: \( a_n = 2n, a_n = 2^n, \) and \( a_n = 2n + 2 \).
   (b) Give a recursive definition of the set of positive integers which are perfect squares.
   (c) Let \( S \) be the set of bit strings defined recursively by \( \epsilon \in S \) and \( 0x1 \in S, 1x0 \in S \) if \( x \in S \), where \( \epsilon \) is the empty string. Find all strings in \( S \) of length not exceeding five, and give an explicit description of the elements of \( S \).

9. Use counting rules to determine the following.
   (a) How many bit strings of length five either begin with two 1s or end with two 0s? How many both begin with two 1s and end with two 0s?
   (b) How many ways can 2 red balls and one blue ball be drawn from an urn containing 7 red and 4
blue balls, if: i) balls are not replaced, and order doesn’t matter; ii) balls are not replaced and you must draw red, blue, red; iii) balls are replaced and you must draw red, blue, red.

(c) How many people must be in a room so that two share the same first digit of their student id?
(d) How long must a sequence of unique integers be to ensure that there is an increasing or decreasing subsequence of at least length 5?

10. Find and solve recurrence relations for each of the following situations.
   (a) The number of ways to build a wall one brick wide and \( n \) inches tall using 1, 2, and 3 inch bricks.
   (b) The number of functions from the set \( \{0, 1, \ldots, n\} \) to the set \( \{0, 1\} \).

11. Use generating functions and/or auxiliary equations to solve the recurrence relations above.

12. Define two relations on the integers \( \{1, 2, 3, 4, 5\} \), \( R_1 \) and \( R_2 \) so that \( R_1 \) is reflexive, anti-symmetric, and not transitive, and \( R_2 \) is irreflexive, not symmetric, but transitive.

13. Using \( R_1 \) and \( R_2 \) above, find \( R_1 \cup R_2 \), \( R_1^{-1} \), \( \overline{R_2} \), \( R_1 \circ R_2 \), and \( R_2 \oplus R_1 \).

14. Find the symmetric closure of \( R_2 \) and the transitive closure of \( R_1 \).

15. Suppose that the following relations are given. Give the sequence of relational operations on the relation required to produce each new relation described below.

<table>
<thead>
<tr>
<th>student_id</th>
<th>name</th>
<th>major</th>
<th>class</th>
<th>club_name</th>
<th>student_id</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>david</td>
<td>math</td>
<td>freshman</td>
<td>math club</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>sarah</td>
<td>comp. sci.</td>
<td>senior</td>
<td>math club</td>
<td>15</td>
</tr>
<tr>
<td>14</td>
<td>david</td>
<td>engineering</td>
<td>junior</td>
<td>cs club</td>
<td>15</td>
</tr>
</tbody>
</table>

(a) The relation containing names and id numbers of all upperclassmen.
(b) The relation containing the names of the clubs to which sarah belongs.

16. Suppose that a class is partitioned into subsets by grade so that \( A = \{x \mid x \text{ gets an A}\} \), and so on.
   (a) Show that this is a partition.
   (b) Define an equivalence relation for this partition.
   (c) What are the equivalence classes of the relation?

17. Construct two graphs, \( G \) and \( H \), both of which are simple undirected graphs with 6 vertices and 10 edges, but which are not isomorphic. Can you do this so that vertices have the same degree?

18. Do either of your graphs above have Hamilton or Euler paths or circuits? Are the connected?

19. Now draw a directed multi-graph \( J \) with 6 vertices and 12 edges. Is your graph strongly and/or weakly connected? What are the components? Find the in and out degrees of each vertex.

20. Give the matrix representations for each of the graphs \( G \), \( H \) and \( J \) above.

21. Give your family tree back two generations. Then, traverse this tree in pre-, in-, and post-order.

22. Write the following logical or arithmetic expressions as trees, and then in pre- and post-fix notation. Evaluate the arithmetic expressions.
   (a) \((\neg p \lor q) \land (r \lor \neg p)\)
   (b) \(+ - 32 \times 52\)
   (c) \((3 + (2 - (4 \times 2^3))) - (3 + (2 + 1)/3)^2\)
   (d) \(253 - *5+\)