Final Exam Review Sheet  
MATH 312 Spring 2003

This exam will cover sections 1.1-2.6, 3.1, 4.1-4.7, 5.1, 6.1, 7.1-7.6, and 8.1-8.3 from your text. You should know general terms, definitions, and theorems from each of these sections. In preparation for the exam, you should review the homework and previous exams, and prepare a one-page 8.5 × 11 paper with formulas to use during the exam.

1. Classifications of differential equations by type, order, linearity, homogeneity, etc.
2. Initial value problems of all types seen in the text, including the application of the two existence and uniqueness theorems seen in the text.
3. Application of D.E.’s to the real-world situations seen in class (i.e. Newton’s Law of Cooling, Spring-Mass Systems, Circuits, etc.).
4. Solving first order equations by separation of variables, exact methods, and linear methods.
5. First-order substitutions including reduction to separable variables, Bernoulli’s equation, etc.
6. Finding the general solution to nth order linear homogeneous differential equations with constant coefficients.
7. Finding particular solutions to non-homogeneous nth order linear D.E.’s via undetermined coefficients, variation of parameter, and/or annihilators.
8. Cauchy-Euler equation solutions.
10. Finding power series solutions about ordinary and regular singular points using appropriate methods.
11. The definition of a Laplace transform, and methods for transforming various functions as seen in the text.
12. Using Laplace transforms and inverse Laplace transforms, including the translations and other shortcuts seen in the text, to assist in solving IVPs and systems of IVPs.

Below is a list of sample problems. Many of these problems are ones you have seen before either in the homework or on previous review sheets. They are reproduced here for convenience.

1. Do the existence-uniqueness theorems guarantee that the following initial-value problems have solutions? Justify your answers.
   (a) \( y' = y \ln(1 - x) \) subject to \( y(2) = 3 \)
   (b) \( y'' + y' \sin x + y \cos x = \sqrt{x^2 + 1} \) for \( y(0) = 0, y'(0) = 1, y''(0) = 2 \).
   (c) \( (1 + x^2)y' = \cos y \) subject to \( y(0) = 0 \)

2. Solve the following initial value problems
   (a) \( xy' - x = 2y^2 \) subject to \( x(1) = 5 \)
   (b) \( y'' - 4y' - 5y = 0 \) subject to \( y(0) = 2, y'(0) = 0 \)
   (c) \( (4y + 2x - 5)dx + (6y + 4x - 1)dy = 0 \) subject to \( y(-1) = 2 \)
   (d) \( y'' + y' + y = t \) subject to \( y(1) = 2, y'(1) = 0 \)

4. Given that \( y_1 = x \sin(\ln x) \) is a solution of \( x^2 y'' - xy' + 2y = 0 \), use reduction of order to find a second solution \( y_2 \) where \( y_1 \) and \( y_2 \) are linearly independent. Then give the general solution to the original equation. Now use the methods appropriate for the Cauchy-Euler equation to find the general solution.

5. Find the general solutions to the following differential equations by any appropriate means.
   (a) \((1 + e^x)y' + e^x + y = 0\)
   (b) \(x^2 y'' + 3xy' = 0\)
   (c) \(x^2 y'' + 9xy' + 8y = x^2\)

6. Consider the differential equation \( x^2 y'' - xy' + y = 0 \) for \( x \) in the interval \((0, \infty)\).
   (a) Show that \( y_1(x) = x \) and \( y_2(x) = x \ln x \) form a fundamental set of solutions.
   (b) Give the general solution of this equation.
   (c) Find constants \( c_1 \) and \( c_2 \) so that \( y(x) = c_1 x + c_2 x \ln x \) satisfies the initial conditions \( y(1) = 3 \) and \( y'(1) = -1 \).

7. Find the eigenvalues and eigenfunctions of the boundary-value problems below.
   (a) \( y'' + \lambda y = 0 \) subject to \( y(0) = 0 \), \( y(\pi) = 0 \)
   (b) \( y'' + 2y' + (\lambda + 1)y = 0 \) subject to \( y(0) = 0 \), \( y(5) = 0 \).

8. Find the first three terms of both series solutions to the following differential equations.
   (a) \( 2xy'' + 2y = 0 \)
   (b) \( y' + x^2 y = 0 \)

9. Use the definition to find the Laplace transform of \( 3t^2 + 4t - 1 \).

10. Write \( f(t) \) in terms of the Heaviside function \( U(t) \), and then find the Laplace transform of \( f(t) \).
    \[
    f(t) = \begin{cases} 
    2 & \text{if } 0 \leq t < 3 \\
    t - 1 & \text{if } 3 \leq t 
    \end{cases}
    \]

11. Find the inverse Laplace transform \( \mathcal{L}^{-1}\left\{\frac{1}{\tau^2 - 5\tau + 6}\right\} \).

12. Solve the IVP \( y'' - 4y' + 4y = t^2 e^{2t} \) subject to \( y(0) = 0 \), \( y'(0) = 0 \) using Laplace transforms.

13. Use Laplace transforms to solve the integral equation below.
    \[
    f(t) + 2 \int_0^t f(\tau) \cos(t - \tau) d\tau = 4e^{-t} + \sin t
    \]

14. Solve the system of differential equations given below using Laplace transforms.
    \[
    \begin{align*}
    \frac{dx}{dt} + 3x + \frac{dy}{dt} &= 1 \\
    \frac{dx}{dt} - x + \frac{dy}{dt} - y &= e^t
    \end{align*}
    \]
    Subject to \( x(0) = 0 \) and \( y(0) = 0 \)
15. Solve each of the following systems of equations using matrix methods.

(a)
\[
\begin{align*}
\frac{dx}{dt} &= 2x + 2y \\
\frac{dy}{dt} &= x + 3y
\end{align*}
\]

(b)
\[
\begin{align*}
\frac{dx}{dt} &= 3x - y - z \\
\frac{dy}{dt} &= x + y - z \\
\frac{dz}{dt} &= x - y + z
\end{align*}
\]

(c)
\[
\begin{align*}
\frac{dx}{dt} &= x - y + 3e^t \\
\frac{dy}{dt} &= x + y + 3e^t
\end{align*}
\]

(d)
\[
\begin{align*}
\frac{dx}{dy} &= y + 1 \\
\frac{dy}{dt} &= -x + \cot t
\end{align*}
\]