Homogeneous Linear Systems

Section 8.2

- Leveraging our Chapter 4 Work
- Eigenvalues and Eigenvectors
- Case I: Distinct Real Eigenvalues
- Case II: Repeated Eigenvalues
- Case III: Complex Eigenvalues
Solution Form

In chapter four we found that a typical solution to a homogeneous linear differential equation had the form:

\[ y = c_1 e^{mx} \]

If we generalize to a homogeneous system of linear differential equations, \( X' = AX \), then what would the “typical” solution vector look like? If we just expand the solution above, we get:

\[
X = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = K e^{\lambda t}
\]
Finding the Solution

In the matrix equations below, we see the consequences of assuming that the solution vector has this form.

\[
\begin{align*}
\left(Ke^{\lambda t}\right)' &= A\left(Ke^{\lambda t}\right) \\
\lambda Ke^{\lambda t} &= A\left(Ke^{\lambda t}\right) \\
\lambda K &= AK \\
AK - \lambda K &= 0 \\
(A - \lambda I)K &= 0
\end{align*}
\]

To get a nontrivial vector \(K\) which solves this expression, we must have \(det(A - \lambda I) = 0\). This is the characteristic equation.
Eigenvalues and Eigenvectors

The $\lambda$s which solve this characteristic equation are called eigenvalues, and the corresponding $K$s are called eigenvectors. When solving a system of equations, we will again run into three distinct cases for our eigenvalues.

- **Case I**: An $n \times n$ matrix $A$ posses $n$ distinct real eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$.

- **Case II**: An $n \times n$ matrix $A$ posses at least one eigenvalue $\lambda$ with multiplicity greater than one.

- **Case III**: An $n \times n$ matrix $A$ posses complex eigenvalues.
Case I: Distinct Real $\lambda$'s

Theorem 8.7

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be $n$ distinct real eigenvalues of the coefficient matrix $A$ for the homogeneous system $X' = AX$, and let $K_1, K_2, \ldots, K_n$ be the corresponding eigenvectors. Then the general solution to the homogeneous system on the entire real line is given by:

$$X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \cdots + c_n K_n e^{\lambda_n t}$$

According then to this theorem, our solution process is to:

- find the characteristic equation,
- solve for eigenvalues,
- find corresponding eigenvectors,
- write in final form.
Examples of Case I

Find the general solution to each system of linear DEs.

\[
\begin{align*}
\frac{dx}{dt} &= 2x + 2y \\
\frac{dy}{dt} &= x + 3y
\end{align*}
\]

\[
X = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^t
\]

\[
\begin{align*}
\frac{dx}{dt} &= 2x - 7y \\
\frac{dy}{dt} &= 5x + 10y + 4z \\
\frac{dz}{dy} &= 5y + 2z
\end{align*}
\]

\[
X = c_1 \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix} e^{7t} + c_3 \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} e^{2t}
\]
Case II: Eigenvalues w/Multiplicity

If an eigenvalue \( \lambda \) has multiplicity \( m < n \), there are 2 possibilities.

- L may yield \( m \) distinct linearly independent eigenvectors. If that is the case, then \( L \) contributes the following to the general solution:
  \[
  c_1 K_1 e^{\lambda t} + c_2 K_2 e^{\lambda t} + \cdots + c_m K_m e^{\lambda t}
  \]

- L may yield only one eigenvector. In this case, \( L \) contributes \( m \) linearly independent solutions of the form:

\[
X_1 = K_{11} e^{\lambda t} \\
X_2 = K_{21} t e^{\lambda t} + K_{22} e^{\lambda t} \\
\vdots \\
X_m = K_{m1} \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + K_{m2} \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \cdots + K_{mm} e^{\lambda t}
\]
Examples of Case II

Find the general solution to the following system of linear DEs.

\[
\begin{align*}
\frac{dx}{dt} &= -6x + 5y \\
\frac{dy}{dt} &= -5x + 4y
\end{align*}
\]

\[
X = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 5 \\ 6 \end{pmatrix} e^{-t} \right]
\]

Case III: While interesting, we will not have time to cover the third in which the characteristic equation yields complex eigenvalues.