MATH 312
Section 4.1: Higher Order Linear Differential Equations

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Outline

1. Homogeneous Linear Differential Equations
   - Existence and Uniqueness
   - Boundary Value Problems
   - Homogeneous Differential Equations
   - Superposition Principle
   - Linear Independence

2. Non-homogeneous Linear Differential Equations
   - Solutions to Non-homogeneous Equations
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3. Conclusion
We now expand our examination to solutions for higher order ($\geq 2$) differential equations. We start with linear DEs.

**nth Order Linear IVPs**

The initial value problem for an $n$th order differential equation asks us to solve

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

subject to the constraints

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

**Note:**

In an initial value problem, we must have information about $y$ and its derivatives at the same point, $x_0$. 
Existence/Uniqueness Theorem

We now have a different existence/uniqueness theorem.

**Theorem 4.1**

Let \( a_n(x), a_{n-1}(x), \ldots, a_0(x) \) and \( g(x) \) be continuous on an open interval \( I \), and let \( a_n(x) \neq 0 \) for every \( x \) in \( I \). Then, if \( x_0 \) is any point in this interval, a solution \( y(x) \) of the IVP below exists and is unique on \( I \).

\[
a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)
\]

subject to the constraints

\[
y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}
\]

**Question:**

How is this similar to the 1st order existence/uniqueness theorem?
Applying the Theorem

Apply this theorem to the following examples.

**Example**

Determine if there is a unique solution to the IVP

\[ 4xy'' + 3y'' + 7x^2y' + y = x \]

subject to

\[ y(1) = 1, \ y'(1) = -1, \ y''(1) = 1 \]

**Example**

Show that \( y = C_1 + C_2 \cos x + C_3 \sin x \) is a solution to \( y''' + y' = 0 \) on the interval \((-\infty, \infty)\) and determine if there is a unique particular solution satisfying

\[ y(\pi) = 0, \ y'(\pi) = 2, \ y''(\pi) = -1 \]
Definition of a BVP

You have probably noticed one of the drawbacks about an IVP is that we must know everything about the solution and its derivatives at a single point.

A boundary value problem (BVP) is one in which we solve

\[
 a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)
\]

subject to \( n - 1 \) conditions on \( y, y', y'', \ldots, y^{(n-1)} \).

These initial conditions are called boundary conditions. A solution \( y \) will satisfy the DE on some interval \( I \) containing the boundary condition points.
A BVP Example

Let’s consider an example boundary value problem.

**Example**

Consider the differential equation $x^2y'' - 5xy' + 8y = 24$ and the family of solutions $C_1x^2 + C_2x^4 + 3$. Determine if a member of the family can be found which satisfies the following boundary conditions.

1. $y(-1) = 0$, $y(1) = 4$
2. $y(-1) = 0$, $y(1) = 0$
3. $y(1) = 3$, $y(2) = 15$

**Existence/Uniqueness**

We do not have a nice existence uniqueness theorem for BVPs at this point.
Homogeneous Linear Differential Equations

Non-homogeneous Linear Differential Equations

Conclusion

Homogeneous Differential Equations

Definition of Homogeneity

Recall that a first order linear equation $\frac{dy}{dx} + P(x)y = 0$ was called homogeneous. We now extend this to higher order DEs.

**Homogeneity**

If the $n$th order linear differential equation below has $g(x) = 0$ we call the equation **homogeneous**. If $g(x) \neq 0$ the equation is **non-homogeneous** and has an associated homogeneous equation in which $g(x)$ does equal 0.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

**Note:**

Unless otherwise stated, we will assume from now on that in a general linear differential equation such as the one above:

- The coefficients $a_i(x)$ and $g(x)$ are continuous
- The leading coefficient $a_n(x) \neq 0$ for each $x$ in the solution interval
Differential Operators

We now turn to a new way to represent differential equations.

Recall from calculus, that differentiation can be represented as:

\[ f'(x) = \frac{df}{dx} = Df(x) \quad \text{and} \quad f^{(n)}(x) = \frac{d^n f}{dx^n} = D^n f(x) \]

Example

Use the linearity of the differential operator \( D \) to rewrite
\( D(C_1 f(x) + C_2 g(x)) \).

Representing a Linear DE

In general, we can represent an \( n \)th order differential equation as:

\[ (a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x))y = g(x) \]

\[ L(y) = g(x) \]
Superposition Principle for Homogeneous DEs

We will begin to examine solutions to $n$th order linear DEs by seeing how solutions can be put together.

**Superposition Principle for Homogeneous DEs**

Let $y_1, y_2, \ldots, y_n$ be solutions of the homogeneous $n$th order differential equation $L(y) = 0$ on an interval $I$. Then, the linear combination $y = C_1y_1 + C_2y_2 + \cdots + C_ky_k$ is a solution on $I$ for arbitrary constants $C_i$.

**Proof**

As $L$ is linear,

\[
L(C_1y_1 + C_2y_2 + \cdots + C_ky_k) = 0
\]
\[
C_1L(y_1) + C_2L(y_2) + \cdots + C_kL(y_k) = 0
\]
\[
C_1(0) + C_2(0) + \cdots + C_k(0) = 0
\]
To see how this works, consider the following example.

**Example**

Consider the differential equation $4y'' - 4y' + y = 0$. Verify that the functions $y_1 = e^{x/2}$ and $y_2 = xe^{x/2}$ are solutions to this differential equation on $(-\infty, \infty)$ and find two new solutions.

**Note:**

Although there are infinitely many solutions which can be constructed from this set, they all have the form of a linear combination of $y_1$ and $y_2$. 
Linearly Independent Functions

It is now natural to consider linear independence.

**Linearly Dependent Functions**

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is **linearly dependent** on an interval $I$ if there are constants $C_1, C_2, \ldots, C_n$ not all zero such that

$$C_1f_1(x) + C_2f_2(x) + \cdots + C_nf_n(x) = 0$$

for all values of $x$ in $I$. If a set of functions does not have this property, it is called **linearly dependent**.

**Example**

Determine if the following sets of functions are linearly independent or linearly independent on $(-\infty, \infty)$.

1. $f_1(x) = 4 + x$, $f_2(x) = 4 + |x|

2. $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = 4x - 3x^2$
The Wronskian

It can be difficult to determine if a set of functions is linearly independent or dependent. To do this, we introduce a new tool.

**Wronskian**

Suppose each of the functions $f_1(x)$, $f_2(x)$, \ldots, $f_n(x)$ has at least $n - 1$ derivatives. Then, the determinant below is called the **Wronskian** of the functions.

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

**Criteria for Linear Independence**

If $y_1$, $y_2$, \ldots, $y_n$ are solutions to an $n$th order homogeneous equation $L(y) = 0$ on some interval $I$, then the set of solutions is linearly independent of $I$ if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every $x \in I$. 
A Fundamental Set of Solutions

At this point, we have not yet stated why we are interested in a linearly independent set of solutions, or indeed that one will even exist!

**Fundamental Sets**

Any set of $n$ linearly independent solutions of the $n$th order homogeneous differential equation $L(y) = 0$ on an interval $I$ is called a **fundamental set of solutions** on that interval.

**Existence**

There exists a fundamental set of solutions for the $n$th order homogeneous differential equation $L(y) = 0$ on an interval $I$.

Having a fundamental set is the key to solving a homogeneous differential equation.
Now, let’s see how we put this together to construct a solution.

The General Solution

If \( \{y_1, y_2, \ldots, y_n\} \) is a fundamental set of solutions of the \( n \)th order homogeneous differential equation \( L(y) = 0 \) on an interval \( I \), then the **general solution** of the equation on that interval is:

\[
y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)
\]

**Example**

Given that \( x^3 \) and \( x^4 \) are both solutions to \( x^2 y'' - 6xy' + 12y = 0 \) on \( (0, \infty) \), find the general solution.
Non-homogeneous Equations

We now turn our attention to non-homogeneous differential equations.

**Particular Solution**

Recall that a particular solution $y_p$ is a function without parameters which solves the non-homogeneous differential equation $L(y_p) = g(x)$.

**General Solution**

Let $y_p$ be any particular solution of the non-homogeneous $n$th order differential equation $L(y) = g(x)$ on some interval $I$, and let \{\(y_1, y_2, \ldots, y_n\)\} be a fundamental set of solutions to the associated homogeneous equation $L(y) = 0$. Then, the general solution to $L(y) = g(x)$ on $I$ is given by:

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) + y_p$$
A Non-homogeneous Example

Consider the following example of this solution form.

Example

Verify that $x^{-\frac{1}{2}}$ and $x^{-1}$ form a fundamental set of solutions to the differential equation $2x^2y'' + 5xy' + y = 0$ on $(0, \infty)$. Then, verify that $\frac{1}{15}x^2 - \frac{1}{6}x$ is a particular solution to $2x^2y'' + 5xy' + y = x^2 - x$ on the same interval. Finally, find the general solution to this second equation.

Note:

Remember to verify that $x^{-\frac{1}{2}}$ and $x^{-1}$ are both solutions and that they are linearly independent.
Superpositioning for Non-homogeneous DEs

We have seen that any particular solution of a non-homogeneous equation can be written in terms of the general solution. What about in the homogeneous case?

Let $y_{p1}, y_{p2}, \ldots, y_{pk}$ be particular solutions of the non-homogeneous differential equations $L(y) = g_i(x)$. Then:

$$y_p(x) = y_{p1}(x) + y_{p2}(x) + \cdots + y_{pk}(x)$$

is a particular solution to the non-homogeneous equation

$$L(y) = g_1(x) + g_2(x) + \cdots + g_n(x)$$

Note:

This superposition principle is different in that we must change the differential equation to which our sum is a solution.
We end with an example of the superposition principle for non-homogeneous differential equations in action.

**Example**

Show that $y_{p1} = 3e^{2x}$ is a particular solution to $y'' - 6y' + 5y = -9e^{2x}$ on the interval $(-\infty, \infty)$.

**Example**

Show that $y_{p2} = x^2 + 3x$ is a particular solution to $y'' - 6y' + 5y = 5x^2 + 3x - 16$ on the interval $(-\infty, \infty)$.

**Example**

Use the examples above to find a particular solution to $y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$.
Important Concepts

Things to Remember from Section 4.1

1. Applying the Existence/Uniqueness Theorem for $n$th order linear DEs
2. Solving IVPs and BVPs
3. Writing homogeneous linear DEs using differential operators
4. Using the superposition principles
5. Verifying linear independence with the Wronskian
6. Verifying a fundamental set of solutions
7. Constructing general solutions