MATH 312
Section 7.5: Dirac Delta & 7.6: Systems of Equations

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Spring Quarter, 2007
Outline

1. Dirac Delta Function
2. Systems of Differential Equations
3. Conclusions
In the real world, many forces act for just a short time. These are called **impulse** forces and can be modeled with the following family of unit impulse functions.

**Unit Impulse Function**

The unit impulse function is actually a family of piecewise defined functions given by:

\[
\delta_a(t - t_0) = \begin{cases} 
0 & 0 \leq t < t_0 - a \\
\frac{1}{2a} & t_0 - a \leq t < t_0 + a \\
0 & t \geq t_0 + a
\end{cases}
\]
The Dirac Delta Function

The limit of these functions as \( a \) goes to zero would give us an instantaneous unit pulse function.

The Dirac delta function is defined by:

\[
\delta(t - t_0) = \lim_{a \to 0} \delta_a(t - t_0)
\]

Properties

The Dirac delta has the following properties, which lead one to realize that it is not really a function.

1. \[
\delta(t - t_0) = \begin{cases} 
\infty & t = t_0 \\
0 & t \neq t_0 
\end{cases}
\]
2. \[
\int_{-\infty}^{\infty} \delta(t - t_0) \, dt = 1
\]
The Laplace Transform of $\delta$

Although not technically a function, we can find the Laplace transform of the $\delta$.

**Theorem 7.11**

For $t_0 > 0$, $\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$.

\[
\delta_a(t - t_0) = \frac{1}{2a} \left[ \mathcal{U}(t - (t_0 - a)) - \mathcal{U}(t - (t_0 + a)) \right]
\]

\[
\mathcal{L}\{\delta_a(t - t_0)\} = e^{-st_0} \left( \frac{e^{sa} - e^{-sa}}{2sa} \right)
\]

\[
\mathcal{L}\{\delta(t - t_0)\} = \lim_{a \to 0} \mathcal{L}\{\delta_a(t - t_0)\} = e^{-st_0}
\]

**Example**

Solve the IVP $y'' + 2y' = \delta(t - 1)$. 
Spring/Mass Example

One of the nice things about the Laplace transform is that it turns a differential equation into an algebraic equation. This allows us to solve systems of equations.

Example

In a coupled spring system, the position of the system is determined by the positions of each individual mass, $x_1$ and $x_2$. To find both $x_1$ and $x_2$ we need two differential equations as shown.

\[
m_1 \frac{d^2 x_1}{dx_1^2} = -kx_1 + k_2(x_2 - x_1)\]

\[
m_2 \frac{d^2 x_2}{dx_2^2} = -k_2(x_2 - x_1)\]
Another practical example in which we need a system of differential equations is in a circuit as shown below.

Example

A circuit may contain several loops, and the current flowing through these loops may be different. If we use differential equations to find that current, then we need a separate equation for each loop.

\[ L \frac{di_2}{dt} + L \frac{di_3}{dt} + R_1 i_2 = E(t) \]

\[ -R_1 \frac{di_2}{dt} + R_2 \frac{di_3}{dt} + \frac{1}{C} i_3 = 0 \]
Finally, we will actually apply the Laplace transform to solve the following problem.

**Example**

Solve the initial value system of differential equations shown below.

\[
\begin{align*}
\frac{dx}{dt} - 4x + \frac{d^3 y}{dt^2} &= 6 \sin t \\
\frac{dx}{dt} + 2x - 2 \frac{d^3 y}{dt^3} &= 0
\end{align*}
\]

subject to:

\[x(0) = 0, \quad y(0) = 0 \quad y'(0) = 0, \quad y''(0) = 0\]
Important Concepts

Things to Remember from Section 7.5 and 7.6

1. Definition and applications of the Dirac Delta function

2. Laplace transforms of the Dirac Delta function

3. Setting-up and solving systems of differential equations including:
   - Spring/Mass Systems
   - Circuits
   - Initial Value Problems