MATH 312
Section 8.1: Systems of First Order Differential Equations

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Outline

1. 1st Order Systems
2. Existence and Uniqueness of Solutions
3. Solution Forms
4. Conclusions
We now turn our attention to solving systems of differential equations. We will start by focusing on first order differential equations.

### 1st Order Systems of Equations

The form of a system of first order differential equations has the form

\[
\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)
\]

\[
\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)
\]

\[
\vdots
\]

\[
\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\]

We assume that the \(a_{ij}(t)\) and the \(f_i(t)\) are continuous on some common interval \(I\). If \(f_i(t) = 0\) for \(1 \leq i \leq n\) then the system is homogeneous.
Matrix Representation

As with systems of algebraic equations, we can use matrices to help represent systems of differential equations.

Matrix Representation

The system above can be represented as

\[
\vec{X}' = A\vec{X} + \vec{F}
\]

where

\[
\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{X}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}
\]
An Example

To see how this works in practice, complete the following example.

**Example**

Write the following system of equations in matrix form.

\[
\begin{align*}
\frac{dx}{dt} &= 3x + 2y - z + \cos t \\
\frac{dy}{dt} &= 2y + 7z + t \\
\frac{dz}{dt} &= -3x + 2z
\end{align*}
\]

**Example**

Is this system of differential equations homogeneous?
Solutions

If we represent systems of differential equations with matrix equations, then our solutions will also be matrices (or vectors).

**Definition 8.1**

A solution vector on an interval $I$ is a column matrix $\mathbf{X}$ whose entries are differentiable functions satisfying $\frac{d\mathbf{X}}{dt} = A\mathbf{X} + \mathbf{F}$ on the interval $I$.

**Example**

Verify that the vector $\mathbf{X}$ shown below is a solution to the given system of differential equations.

\[
\begin{bmatrix}
-\frac{e^{-3t/2}}{2} \\
\frac{e^{-3t/2}}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
-1 & \frac{1}{4} \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]
Initial Value Problems

What about initial value problems for systems of linear equations?

Initial Value Problem
An initial value problem for a system of 1st order differential equations asks to solve $\mathbf{X}' = A\mathbf{X} + \mathbf{F}$ subject to $\mathbf{X}(t_0) = \mathbf{X}_0$.

Theorem 8.1
The initial value problem $\mathbf{X}' = A\mathbf{X} + \mathbf{F}$ subject to $\mathbf{X}(t_0) = \mathbf{X}_0$ has a unique solution on an interval $I$ containing $t_0$ if the entries in $A$ and $\mathbf{F}$ are continuous functions on the interval $I$.

Example
On what interval would an initial value problem with the following system of equations be guaranteed a unique solution?

\[
\frac{dx}{dt} = 3x + y + \ln t \quad \frac{dy}{dt} = x - 7y + \frac{1}{t - 3}
\]
Superpositioning Principle

So how do we construct a solution to a system of 1st order linear equations?

**Superpositioning**

Let $\vec{X}_1, \vec{X}_2, \cdots, \vec{X}_n$ be solution vectors of a homogeneous system $\vec{X} = A\vec{X}$ on an interval $I$. Then the linear combination $c_1\vec{X}_1 + c_2\vec{X}_2 + \cdots + c_n\vec{X}_n$ is also a solution on $I$ for constants $c_1, c_2, \ldots c_n$.

**Example**

Verify that $\vec{X}_1$ and $\vec{X}_2$ shown below are solutions to the system of equations and find a 3rd solution.

$$\vec{X}' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \vec{X}$$

$$\vec{X}_1 = \begin{bmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{bmatrix}$$

$$\vec{X}_2 = \begin{bmatrix} 0 \\ e^t \\ 0 \end{bmatrix}$$
Linear (In)Dependence

As was the case with a single differential equation, we need to find solutions which are independent of each other.

**Definition 8.2**

Solution vectors $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_k$ are called **linearly dependent** on the solution interval $I$ if there are constants $c_1, c_2, \ldots, c_k$ not all zero, for which $c_1\vec{X}_1 + c_2\vec{X}_2 + \cdots + c_k\vec{X}_k = 0$. If the set is not linearly dependent, it is **linearly independent**.

**Theorem 8.3**

Let $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$ be a set of solution vectors for a homogeneous system $\vec{X}' = A\vec{X}$ on an interval $I$. Then the set is linearly independent on $I$ if and only if the Wronskian, $W(\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n)$, is not zero for every $t$ in $I$. 
Fundamental Sets of Solutions

As was the case with a single differential equation, we can define a fundamental set of solutions.

**Fundamental Set of Solutions**

Any set \( \vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n \) of \( n \) linearly independent solution vectors to a homogeneous system \( \vec{X}' = A\vec{X} \) of \( n \) equations is a fundamental set of solutions.

**Example**

Could the following vectors be a fundamental set of solutions on \((-\infty, \infty)\)?

\[
\vec{X}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\
\vec{X}_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \\
\vec{X}_3 = \begin{bmatrix} 3 \\ -6 \\ 12 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}
\]
Existence and Form of Solutions

The following theorems tell us how to construct solutions.

**Theorem 8.4**
There exists a fundamental set of solutions for the homogeneous system $\mathbf{X}' = A\mathbf{X}$ on an interval $I$.

**Theorem 8.5**
Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ be a fundamental set of solutions to the homogeneous system $\mathbf{X}' = A\mathbf{X}$ on an interval $I$. Then the general solution of the system on the interval is

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + \cdots + c_k \mathbf{X}_k$$

where the $c_i$, $i = 1, 2, \ldots, k$ are arbitrary constants.
Particular Solutions

We end by looking at non-homogeneous systems of differential equations.

**Theorem 8.6**

Let $\vec{X}_p$ be a particular solution to the nonhomogeneous system $\vec{X}' = A\vec{X} + \vec{F}$ on $I$, and $\vec{X}_c$ the general solution to the associated homogeneous equation. Then $\vec{X} = \vec{X}_c + \vec{X}_p$ is the general solution to the nonhomogeneous system.

**Example**

Show that the general solution to

$$\vec{X}' = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \vec{X} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 4 \\ -6 \end{bmatrix} t + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

on the interval $(-\infty, \infty)$ is

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ -1 - \sqrt{2} \end{bmatrix} e^{\sqrt{2}t} + c_2 \begin{bmatrix} 1 \\ -1 + \sqrt{2} \end{bmatrix} e^{-\sqrt{2}t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
Important Concepts

Things to Remember from Section 8.1

1. Writing systems of differential equations in matrix form.
2. Identifying homogeneous systems of differential equations.
4. Showing solutions sets are independent or fundamental sets.