Exam I Review Sheet
MATH 312, Spring 2007

This exam will cover sections 1.1-2.5, 3.1 and 3.2 in your text. You should know general terms, definitions, and theorems from each of these sections, review the homework given for these sections, and pay particular attention to the subjects and practice problems mentioned below.

1. Basic terms and definitions, including type, order, and linearity.
2. Verification of both implicit and explicit solutions to D.E.’s.
3. Initial value problems and questions of existence and uniqueness, including the statement and application of Theorem 1.1.
4. Formulation of D.E.’s for basic mathematical models.
5. Using direction fields, and critical points of autonomous 1st order D.E.’s to estimate solution curves.
6. Definition of and solutions to homogeneous and non-homogeneous linear first order D.E.’s.
8. Substitutions for homogeneous equations, Bernoulli’s equation, and reduction to separable variables.
9. Solutions to first order linear mathematical models.
10. Solutions to first order non-linear mathematical models.

Below is a list of sample problems. This list is not all-inclusive, but does represent the types of problems you will see on the exam.

1. Show that \( y = x^2 + x^2 \ln x \) is a solution of the differential equation \( x^2 y'' - 3xy' + 4y = 0 \). Give the order of this differential equation.

2. State the existence-uniqueness theorem (Theorem 1.1) for first order initial-value problems.

3. Does the existence-uniqueness theorem guarantee that the following initial-value problems have solutions? Justify your answers.
   (a) \((1 - y)y' = \sqrt{x} \sin y\) subject to \(y(1) = 0\)
   (b) \(yy' = \sin x\) subject to \(y(0) = 1\)
   (c) \(yy' = \sin x\) subject to \(y(1) = 0\)
   (d) \(y' = y \ln(1 - x)\) subject to \(y(2) = 3\)
   (e) \((1 + x^2)y' = \cos y\) subject to \(y(0) = 0\)

4. Find values of \(m\) so that \(y = e^{mx}\) is a solution of the differential equation \(y'' + 10y' + 25y = 0\)

5. Find a value of \(k\) so that \(y = \sin kx\) is a solution of the differential equation \(y'' + 25y = 0\)

6. Solve the following initial-value problem (implicit solutions okay):
   \((y \cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0\) subject to \(y(0) = 1\)

7. Determine a region in the xy-plane for which the differential equations below have unique solutions passing through the point \((x_0, y_0)\). Classify each equation by linearity and order.
   (a) \(x \frac{dy}{dx} = y^2\)
   (b) \(y \frac{dy}{dx} = \sqrt{x + 1}\)
8. Consider the differential equation \( \frac{dy}{dx} = y^2 - 1 \).
   (a) Sketch a direction field for this differential equation.
   (b) Sketch a graph of the solution that passes through the point \((1, 0)\).
   (c) What can be said about the point \((0, 1)\)?

9. Solve the following initial value problems
   (a) \( \frac{dy}{dx} + y \sin x = 2 \sin x \) subject to \( y(0) = 0 \)
   (b) \( (4y + 2x - 5)dx + (6y + 4x - 1)dy = 0 \) subject to \( y(-1) = 2 \)
   (c) \( \frac{dy}{dx} + y = x \) subject to \( y(0) = 2 \)
   (d) \( \frac{dy}{dx} = \sqrt{x + 1} \) subject to \( y(0) = -1 \)
   (e) \( \frac{dy}{dx} = 5(y^2 - y) \) subject to \( y(0) = 2 \)

10. Which of the following functions are homogeneous? Justify your answers.
    (a) \( f(x, y) = x^2 + xy + y^2 \)
    (b) \( f(x, y) = \ln y - \ln x + 3 \)
    (c) \( f(x, y) = x^2 + 1 \)

11. Which of the following differential equations are exact? Justify your answers and solve all exact or
    homogeneous differential equations.
    (a) \((x^3 + y^3)dx = -3xy^2dy\)
    (b) \(\frac{dx}{y} - \frac{x^2}{y^2}dy = 0\)
    (c) \(xy' = 2xe^x - y + 6x^2\)

12. Solve the following mathematical models.
    (a) According to Newton's law of cooling, the rate at which a body cools is proportional to the
        difference between the temperature of the body and the temperature of the surrounding medium.
        A roast turkey is taken from an oven when its temperature has reached 180 degrees F and is
        placed on a table in a room where the temperature is 75 degrees F. If the temperature of the
        turkey is 150 degrees F after half an hour, what will the temperature be after 45 minutes? When
        will the turkey have cooled to 100 degrees F?
    (b) A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram
        of salt per liter is then pumped into the tank at a rate of 4 liters per minute, and the well-mixed
        solution is then pumped out at the same rate. Find the number \(A(t)\) of grams of salt in the tank
        at any time \(t\).

13. Solve the following differential equations.
    (a) \(xydy = (x^2 + y^2)dx\)
    (b) \(y' + y^2 \sin x = 0\)

14. Find an integrating factor for each of the following differential equations, and then find the solution.
    (a) \(y' + 2xy = x\) subject to \(y(1) = 0\)
    (b) \(y' + \frac{1}{x+1}y = 2\) subject to \(y(1) = 1\)
    (c) \((1 + e^x)y' + e^x + y = 0\)