Solving Simple Differential Equations

Section 2.2: Separable Variables

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1. Solving Simple Differential Equations
2. Separable Variables
3. Revisiting IVPs and Models
4. Conclusion
In our quest for solution methods, we start with finding solutions to certain first order differential equations.

Example

Solve the differential equation \( \frac{dy}{dx} = x + 1 - \sin x \).

\[
\int \frac{dy}{dx} \, dx = \int (x + 1 - \sin x) \, dx
\]

\[
y = \frac{x^2}{2} + x + \cos x + C
\]

When can we solve a differential equation in this fashion?
When Can we Solve with Integration Alone?

Differential equations for which a solution can be found by simple integration are called separable.

**Separable DEs**

A first order DE of the form $\frac{dy}{dx} = f(x, y)$ is said to be separable, or to have *separable variables* if we can rewrite $f(x, y)$ as $f(x, y) = g(x)h(y)$.

**Example**

Which of the differential equations below are separable?

- $\frac{dy}{dx} = x \sin y + 3xe^y$
- $\frac{dy}{dx} = e^x + e^y$
The general solution process for a separable differential equation is shown below.

### Solution Process

\[
\frac{dy}{dx} = g(x)h(y)
\]

\[
\frac{1}{h(y)} \frac{dy}{dx} = g(x)
\]

\[
\int \frac{1}{h(y)} \, dy = \int g(x) \, dx
\]

\[
H(y) = G(x) + C \quad \text{where} \quad H'(y) = \frac{1}{h(y)} \quad \text{and} \quad G'(x) = g(x)
\]
Example Problems

We now practice solving separable differential equations using this procedure.

Example

Solve each differential equation using separation of variables.

1. \( \frac{dy}{dx} + 2xy = 0 \)
2. \( \frac{dy}{dx} = x \sqrt{1 - y^2} \)

Note:

Notice that the constant function \( y = 0 \) is a solution to the first example and \( y = 1 \) is a solution to the second.
As we just noticed, there are often constant solutions to a differential equation.

**Singular Solutions**

A singular solution to a DE is a solution which is not a part of the family of solutions obtained by the solution process.

**Singular Solutions to Separable DEs**

In the separation of variable process, we may lose singular solutions of the form $y = c$ (which would mean $\frac{dy}{dx} = f(x, c) = 0$) if $h(y) = h(c) = 0$ making $\frac{1}{h(y)}$ undefined.
An Initial Value Example

For suitable initial value problems, the method of separation of variables can be used to find a solution.

Example

Solve the initial value problem \( \frac{dy}{dx} = \frac{y^2 - 1}{x^2 - 1} \) subject to \( y(2) = 2 \).

Remember!

We must check for singular solutions. In our case, \( y = \pm 1 \) are singular solutions to the DE, but not to the IVP.
An Implicit Solution to an IVP

It is not always the case that our solutions will be explicit. Consider the following IVP.

**Example**

Find both an implicit and explicit solution to the IVP:

\[(1 + x^4)dy + x(1 + 4y^2)dx = 0 \quad \text{subject to} \quad y(1) = 0\]

1. Separate the variables
2. Integrate
3. Solve for constant of integration, yielding implicit solution
4. Solve for \(y\) to find explicit solution
A Solution to a Mixture Model Problem

Some of the first order models seen in section 1.3 can now be solved.

**Example**
Suppose that a large mixing tank initially holds 300 gallons of water in which 10 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is then pumped out at the same rate. Determine a differential equation for the amount of salt $A(t)$ in the tank at time $t$. What is $A(0)$?

The example yields the model:

$$\frac{dA}{dt} = \frac{A}{100} \quad \text{subject to} \quad A(0) = 10$$
Important Concepts

Things to Remember from Section 2.2

1. Identifying DEs with separable variables
2. Solving differential equations with separable variables
3. Solving separable IVPs