Outline

1. Solution by Substitution
2. Homogeneous Differential Equations
3. Bernoulli’s Equation
4. Reduction to Separation of Variables
5. Conclusion
A Motivating Example

In this last section of chapter 2, we introduce no new methods of solving DEs but rather look at ways to reduce a DE to a type we already know how to solve.

Example

Solve the following differential equation.

\[(y^2 + yx) \, dx + x^2 \, dy = 0\]

Your first impulse might be to try exact solution methods. However:

- The equation is not exact.
- \(\frac{M_y - N_x}{N} = \frac{2y - x}{x^2}\) and \(\frac{N_x - M_y}{M} = \frac{x - 2y}{y^2 + yx}\).
- Finally, \(\frac{dy}{dx} = -\frac{y^2 + yx}{x^2}\) is neither separable nor linear.
What is a Homogeneous DE? (this time. . . )

Unfortunately, the name for differential equations in which our first substitution works has already been used in this class.

**Definition**

If \( f(x, y) \) is a function such that \( f(tx, ty) = t^\alpha f(x, y) \) for some real number \( \alpha \), then \( f \) is a **homogeneous function of degree** \( \alpha \).

**Definition**

If \( M(x, y) \, dx + N(x, y) \, dy = 0 \) is a first order differential equation in differential form, then it is called **homogeneous** if both \( M \) and \( N \) are homogeneous functions of the same degree.
Identifying Homogeneous DEs

Let's examine several examples, including our motivating example.

Example

Is the following differential equation homogeneous? **No.**

\[ (3x^2 + 1) \, dx + (3y^2 - 4x) \, dy = 0 \]

Example

Is the following differential equation homogeneous? **No.**

\[ (3x^2 + y^2) \, dx + (xy^2) \, dy = 0 \]

Example

Is the following differential equation homogeneous? **Yes!**

\[ (y^2 + yx) \, dx + x^2 \, dy = 0 \]
Solution Procedure

The reason homogeneous differential equations are of interest is that they allow us to make the equation separable by substitution.

Solving A Homogeneous DE

To solve a homogeneous differential equation of the form $M(x, y) \, dx + N(x, y) \, dy = 0$ let $y = ux$.

$M(x, ux) \, dx + N(x, ux)(u \, dx + x \, du) = 0$

$x^\alpha M(1, u) \, dx + x^\alpha N(1, u)(u \, dx + x \, du) = 0$

$[M(1, u) + uN(1, u)] \, dx + xN(1, u) \, du = 0$

$\frac{dx}{x} = \frac{-N(1, u) \, du}{M(1, u) + uN(1, u)}$ separable!
Examples

We now apply this procedure to several examples.

Example
Solve the homogeneous differential equation

$$(y^2 + yx) \, dx + x^2 \, dy = 0$$

$$x^2 y = C(y + 2x)$$

Example
Solve the initial value problem

$$y \, dx + x(ln x - ln y - 1) \, dy = 0 \quad \text{subject to} \quad y(1) = -e$$

$$y \ln \left| \frac{x}{y} \right| = e$$
Bernoulli’s Equation and Linear DEs

Another substitution leads to the solution of what is called Bernoulli’s Equation (actually a family of equations) by linearity.

**Bernoulli’s Equation**

An equation of the form below is called Bernoulli’s Equation and is non-linear when $n \neq 0, 1$.

\[
\frac{dy}{dx} + P(x)y = f(x)y^n
\]

**Solving Bernoulli’s Equation**

In order to reduce a Bernoulli’s Equation to a linear equation, substitute $u = y^{1-n}$.
Justifying the Substitution

So why does substitution $u = y^{1-n}$ into $\frac{dy}{dx} + P(x)y = f(x)y^n$ help us solve the DE?

$$y = u^{\frac{1}{1-n}} \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{1-n} u^{\frac{n}{1-n}} \frac{du}{dx}$$

$$\frac{1}{1-n} u^{\frac{n}{1-n}} \frac{du}{dx} + P(x) u^{\frac{1}{1-n}} = f(x) u^{\frac{n}{1-n}}$$

$$\frac{du}{dx} + (1-n)P(x)u = f(x) \quad \text{(multiply by } (1-n)u^{\frac{-n}{1-n}})$$

Solve for $u$ using linearity and then let $y^{1-n} = u(x)$. 
An Example

In general, we will apply this procedure to each distinct example to avoid symbol confusion.

Example

Solve the differential equation

\[ \frac{dy}{dx} - y = e^x y^2 \]

Let \( u = y^{1-2} = y^{-1} \) and \( y = u^{-1} \) and \( \frac{dy}{dx} = -u^{-2} \frac{du}{dx} \)

\[-u^{-2} \frac{du}{dx} - u^{-1} = e^x u^{-2} \quad \Rightarrow \quad \frac{du}{dx} + u = -e^x\]

\[ y^{-1} = -\frac{1}{2} e^x + Ce^{-x} \]
Our final substitution is one with a very specific form.

Reduction to Separable

A differential equation of the form below can always be reduced to a separable equation (provided $B \neq 0$ by the substitution $u = Ax + By + C$.

$$\frac{dy}{dx} = f(Ax + By + C)$$

$$u = Ax + By + C \quad \text{and} \quad \frac{du}{dx} = A + B \frac{dy}{dx}$$

$$\frac{1}{B} \left( \frac{du}{dx} - A \right) = f(u)$$

$$\frac{du}{Bf(u) + A} = dx$$
An Example

As with Bernoulli’s Equation, do not memorize the formulas, just follow the procedure.

**Example**

Solve the differential equation

\[
\frac{dy}{dx} = \sin(x + y)
\]

\[u = x + y\]

and

\[
\frac{dy}{dx} = \frac{du}{dx} - 1
\]

\[
\frac{du}{\sin(u) + 1} = dx \Rightarrow -\frac{\sin(u) - 1}{\cos^2 u} \, du = dx
\]

\[-\sec(x + y) + \tan(x + y) = x + C\]
Important Concepts

Things to Remember from Section 2.5

1. Identifying homogeneous functions and differential equations.
2. Solving homogeneous differential equations by substitution.
3. Solving Bernoulli’s equation by substitution.
4. Reduction to separable.