Outline

1. The Logistic Growth Model
2. Logistic vs. Exponential Growth
3. Chemical Reactions
4. Conclusion
A New Population Growth Model

Our first non-linear model is a refinement of the previously studied population growth model.

Exponential Growth Model

In the exponential growth model, our key assumption is that the relative growth rate of the population is constant.

\[
\frac{dP}{dt} \frac{1}{P} = \lambda
\]

Logistic Growth Model

In the logistic growth model, our key assumption is that the relative growth rate of the population is a linear function of \( P \).

\[
\frac{dP}{dt} \frac{1}{P} = a - bP
\]
A quick expansion of the logistic growth equation shows that this is a non-linear differential equation.

Example
Solve the logistic growth equation

\[
\frac{dP}{dt} = \frac{a - bP}{P}
\]

\[
\frac{dP}{P(a - bP)} = dt
\]

\[
\left(\frac{1}{a} + \frac{b/a}{a - bP}\right) dP = dt
\]

\[
\frac{1}{a} \ln \left|\frac{P}{a - bP}\right| = t + C_1
\]

\[
P(t) = \frac{aC}{bC + e^{-at}}
\]
Properties of Logistic Growth

What are the underlying assumptions of the logistic growth model?

**Carrying Capacity**

Logistic growth models assume that a limited amount of resources places a **carrying capacity**, or upper bound, on the population. Observe that

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{aC}{bC + e^{-at}} = \frac{a}{b}$$

**Point of Inflection**

The second derivative test gives a point of inflection for $P(t)$ which can help us graph this model.

$$\frac{d^2P}{dt^2} = 2b^2P \left( P - \frac{a}{b} \right) \left( P - \frac{a}{2b} \right)$$
A Logistic Growth Example

We now examine a particular logistic growth example.

Example

A rabbit population of 50 is transplanted to an isolated valley. The rabbits population has an initial growth rate of 0.50 rabbits per month. If the valley can only support a population of 5000 rabbits, find the logistic function modeling this situation and determine when the valley will reach 90% of its capacity.

\[
\frac{dP}{dt} = P \left(0.50 - \frac{P}{10,000}\right)
\]

\[
P(t) = \frac{50.51}{0.01 + e^{-0.5t}}
\]

\[
P(t) = 0.90(5000) = 4500 \Rightarrow t \approx 13.4
\]
A Population Growth Example

Let’s compare the logistic growth model with the exponential model on a real-world example.

Example

Use U.S. Census data shown to construct both an exponential and logistic growth model for the U.S. Population. Then use Excel to compare these two models and compute the error and percentage error for each year.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
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<tbody>
<tr>
<td>1790</td>
<td>3.92921</td>
</tr>
<tr>
<td>1800</td>
<td>5.30848</td>
</tr>
<tr>
<td>1810</td>
<td>7.23988</td>
</tr>
<tr>
<td>1820</td>
<td>9.63846</td>
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<td>17.0633</td>
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<td>1850</td>
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<td>1860</td>
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<tr>
<td>1870</td>
<td>38.5584</td>
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<tr>
<td>1880</td>
<td>50.1892</td>
</tr>
<tr>
<td>1890</td>
<td>62.9798</td>
</tr>
<tr>
<td>1900</td>
<td>75.2122</td>
</tr>
<tr>
<td>1910</td>
<td>92.2285</td>
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<tr>
<td>1920</td>
<td>106.022</td>
</tr>
<tr>
<td>1930</td>
<td>123.203</td>
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<tr>
<td>1940</td>
<td>132.165</td>
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<tr>
<td>1950</td>
<td>151.325</td>
</tr>
<tr>
<td>1960</td>
<td>178.323</td>
</tr>
<tr>
<td>1970</td>
<td>203.302</td>
</tr>
<tr>
<td>1980</td>
<td>226.562</td>
</tr>
</tbody>
</table>
Solutions

Let’s look at the two growth models we have seen.

### Exponential Model

Using the model \( \frac{dP}{dt} = \lambda P \) with \( P(0) = 3.92921 \) and \( P(190) = 226.562 \) yields:

\[
P(t) = 3.92921 e^{0.0213t}
\]

### Logistic Model

Using the model \( \frac{dP}{dt} = P(a - bP) \) with \( P(0) = 3.92921, P(90) = 50.18920 \) and \( P(180) = 203.302 \) yields:

\[
P(t) = \frac{256.853}{1 + 64.37 e^{-0.0305t}}
\]

### Comparison

Now, use this spreadsheet to compare these models.
A Chemical Reaction Model

Our second non-linear model involves chemical reactions.

**Chemical Reactions**

Suppose that a chemical reaction between $a$ grams of chemical $A$ and $b$ grams of chemical $B$ forms chemical $C$ consisting of $M$ parts chemical $A$ and $N$ parts chemical $B$. If $x(t)$ is the number of grams of chemical $C$ produced at time $t$, then the number of grams of $A$ and $B$ remaining at time $t$ is:

$$a - \frac{M}{M+N}X \quad b - \frac{N}{M+N}X$$

When no temperature change is involved, the rate of the reaction is assumed to be directly proportional to the product of the amounts of chemicals $A$ and $B$ remaining. Thus:

$$\frac{dX}{dt} = k_1 \left( a - \frac{M}{M+N}X \right) \left( b - \frac{N}{M+N}X \right)$$
In order to solve this model, it helps to rewrite it in a simpler form.

**Simplifying the Model**

\[
\frac{dX}{dt} = k_1 \left( a - \frac{M}{M+N} X \right) \left( b - \frac{N}{M+N} X \right)
\]

\[
\frac{dX}{dt} = k_1 \left( \frac{M}{M+N} \right) \left( \frac{N}{M+N} \right) \left( a - \frac{M}{M+N} X \right) \left( b - \frac{N}{M+N} X \right)
\]

\[
\frac{dX}{dt} = k (\alpha - X) (\beta - X)
\]
Chemical Reaction Example

Now, let's look at a chemical reaction example.

**Example**

Suppose that chemicals $A$ and $B$ combine to form chemical $C$ in such a way that 3 grams of $A$ and 2 grams of $B$ are used for each gram of $C$. You observe that in 20 minutes, 50 grams of $C$ has been produced. Determine the amount of $C$ which will eventually be produced from this reaction if you started with 70 grams of $A$ and 61 grams of $B$.

\[
\frac{dX}{dt} = k_1 \left( 70 - \frac{3}{5}X \right) \left( 61 - \frac{2}{5}X \right)
\]

\[
\frac{dX}{dt} = k \left( 116.67 - X \right) \left( 152.5 - X \right)
\]
Chemical Reaction Example, continued

Continuing with this computation.

\[
\frac{dX}{(116.67 - X)(152.5 - X)} = k \, dt
\]

\[
\frac{152.5 - X}{116.67 - X} = Ce^{3.57kt}
\]

\[
X(t) = \frac{152.84e^{3.57kt} - 152.5}{1.31e^{3.57kt} - 1}
\]

\[
X(t) = \frac{152.84e^{0.008t} - 152.5}{1.31e^{0.008t} - 1}
\]

As \( t \to \infty \) we get \( X(t) \to 116.67 \)
### Important Concepts

<table>
<thead>
<tr>
<th></th>
<th>Things to Remember from Section 3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Setting up and solving non-linear models</td>
</tr>
<tr>
<td>2</td>
<td>Working with the logistic growth model</td>
</tr>
<tr>
<td>3</td>
<td>Solving chemical reaction problems</td>
</tr>
</tbody>
</table>