MATH 312
Section 4.5: Undetermined Coefficients

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We now turn our attention to finding a particular solution to a non-homogeneous linear DE. The tool we will use for this is the differential operator.

Recall that an $n$th order non-homogeneous linear differential equation (with constant coefficients) can be represented as:

$$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 D y + a_0 y = g(x)$$

We often rewrite by “factoring” out the $y$ and write $L(y) = g(x)$.

What is $D^n$?

Recall that the differential operator $D^n y = \frac{d^n y}{dx^n}$.
Let’s practice working with differential operators.

**Example**

Rewrite the differential equation below using linear operators in the form $L(y) = g(x)$.

$$y''' + 2y'' - 9y' - 18y = xe^{-x}$$

You should have found:

$$(D^3 + 2D^2 - 9D - 18)(y) = xe^{-x}$$

If we cheat and think of $D$ as a real number, we could write:

$$(D + 3)(D - 3)(D + 2)(y) = xe^{-x}$$
Justifying the Factorization

Does it really make sense to “factor” a differential operator?

**Example**

Show that applying the differential operators

$$(D + 3)(D - 3)(D + 2)$$

and

$$(D^3 + 2D^2 - 9D - 18)$$

to $y$ produces the same expression.

$$(D + 3)(D - 3)(D + 2)y = (D + 3)(D - 3)(y' + 2y)$$

$$= (D + 3)(y'' + 2y' - 3y' - 6y)$$

$$= y''' + 2y'' - 9y' - 18y$$

$$= (D^3 + 2D^2 - 9D - 18)y$$
Annihilating $g(x)$

Why would we want to rewrite a differential equation using factored differential operators?

**Example**

Suppose that $L(y) = 10x^3 - 2x$ is our differential equation. Let $L_2$ be the differential operator $D^4$. Apply $L_2$ to both sides of this differential equation.

You should have found that $10x^3 - 2x$ was annihilated by $D^4$ producing a new, higher order, homogeneous equation.

**Annihilating**

If there is a differential operator $L_2$ such that $L_2(g(x)) = 0$, then we can annihilate $g(x)$ from the differential equation $L(y) = g(x)$ producing a homogeneous equation $L_2(L(y)) = 0$. 
Annihilating Polynomials

We will now start a collection of annihilating differential operators based on the type of the function $g(x)$.

Annihilating Polynomials

Let $g(x) = C_1 + C_2x + \cdots + C_nx^{n-1}$ be a polynomial. Then, the differential operator $D^n$ will annihilate $g(x)$.

Example

Use induction on $n$, the degree of $g(x)$, to prove that this annihilator works.

Example

Find an annihilator for each $g(x)$.

1. $3x^2 - 3x + 1$
2. $5x + 6x^5 - 3x^2$
Annihilating Exponentials

The next class of functions we will attempt to annihilate is the exponential function.

**Annihilating Exponentials**

Let \( g(x) = C_1 e^{\alpha x} + C_2 xe^{\alpha x} + \cdots + C_n x^{n-1} e^{\alpha x} \) be an exponential function. Then, the differential operator \((D - \alpha)^n\) will annihilate \( g(x) \).

**Example**

Use induction on \( n \) to prove that this annihilator works.

**Example**

Find an annihilator for each \( g(x) \).

1. \( xe^{2x} \)
2. \( e^{7x} - x^5 e^{7x} \)
Finally, we deal with the trigonometric functions \( \sin x \) and \( \cos x \).

### Trigonometric Functions

Let \( g(x) = Cx^{n-1}e^{\alpha x} \cos \beta x \) or \( Cx^{n-1}e^{\alpha x} \sin \beta x \). Then, the differential operator \( [D^2 - 2\alpha D + (\alpha^2 + \beta^2)] \) annihilates \( g(x) \).

### Example

While this is certainly true in all cases, we prove only that
\[
[D^2 - 2\alpha D + (\alpha^2 + \beta^2)] (e^{\alpha x} \cos \beta x) = 0.
\]

### Example

Find an annihilator for each \( g(x) \).

1. \( 2xe^x \sin 4x \)
2. \( 2 \cos 8x - 5 \sin 8x \)
Combining Annihilators

What if our function \( g(x) \) is a sum or difference of these types of functions?

**Working with Differential Operators**

Show that if \( L_1 \) annihilates \( g_1(x) \) and \( L_2 \) annihilates \( g_2(x) \) then \( L_1 L_2 \) annihilates \( g_1(x) \pm g_2(x) \).

**Example**

Find differential operators which will annihilate each of the following functions.

1. \( g(x) = 3 + e^x \cos 2x \)
2. \( g(x) = 3x^2 - \cos x \)
We summarize our annihilator findings below in preparation for the next section.

<table>
<thead>
<tr>
<th>Function</th>
<th>Annihilator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$$x^{n-1}$$</td>
<td>$$D^n$$</td>
</tr>
<tr>
<td>$$x^{n-1}e^{\alpha x}$$</td>
<td>$$(D - \alpha)^n$$</td>
</tr>
<tr>
<td>$$x^{n-1}e^{\alpha x} \cos \beta x$$, $$x^{n-1}e^{\alpha x} \sin \beta x$$</td>
<td>$$\left[D^2 - 2\alpha D + (\alpha^2 + \beta^2)\right]^n$$</td>
</tr>
</tbody>
</table>
Why Annihilators are of Interest

Recall that the reason we are interested in annihilators is that they can turn non-homogeneous differential equations into homogeneous differential equations.

Annihilating

Consider the linear non-homogeneous differential equation $L(y) = g(x)$. If there is a differential operator $L_2$ such that $L_2(g(x)) = 0$, then we can annihilate $g(x)$ and produce the homogeneous equation $L_2(L(y)) = 0$.

We now need to connect a solution to $L_2(L(y)) = 0$ with the solutions to $L(y) = g(x)$. 
Solution Step 1: Solve the Homogeneous

During the next several sides, we will work through the following example.

**Example**

Solve the differential equation

\[ y'' + 4y' + 4y = 2x + 6 \]

**Step 1:**

First we solve the associated homogeneous differential equation:

\[ y'' + 4y' + 4y = 0 \]

\[ y_c = C_1 e^{-2x} + C_2 xe^{-2x} \]
The next step in our solution process is to find an annihilator.

**Step 2:**

We now find an annihilator for $g(x) = 2x + 6$ and apply it.

\[ \begin{align*}
D^2 &\quad \text{annihilates} \quad 2x + 6 \\
D^2(D^2 + 4D + 4) &= D^2(2x + 6) \\
(D^4 + 4D^3 + 4D^2)y &= 0 \\
y &= C_1 + C_2x + C_3e^{-2x} + C_4xe^{-2x} \\
\end{align*} \]
Solution Step 3: Remove $y_c$ and Solve for $y_p$

Next, we remove $y_c$ from our solution to $L_2(L(y)) = 0$ and determine the appropriate coefficients for the remaining pieces.

**Step 3:**

The remaining portion of the solution to $L_2(L(y)) = 0$ is

$$y_p = A + Bx$$

$$(0) + 4(B) + 4(A + Bx) = 2x + 6$$

$$4B + 4A = 6 \quad \text{and} \quad 4B = 2$$

$$y_p = 1 + \frac{1}{2}x$$
Summary of Solution Process

This results in the following general solution.

Example

The solution to \( y'' + 4y' + 4y = 2x + 6 \) is:

\[
y = C_1 e^{-2x} + C_2 xe^{-2x} + \frac{1}{2}x + 1
\]

Solution Procedure:

To solve a differential equation \( L(y) = g(x) \) follow this process.

- Solve the associated homogeneous equation \( L(y) = 0 \) to find \( y_c \).
- Find an annihilator \( L_1 \) for \( g(x) \) and apply it to both sides. Solve the new equation \( L_1(L(y)) = 0 \).
- Delete from the solution obtained in the previous stem all terms which are in \( y_c \) and use undetermined coefficients to find \( y_p \).
- The general solution is now \( y = y_c + y_p \).
Examples

While the steps in this process are relatively straightforward, it can take some work to solve an equation using this method.

Example

Find the general solution to the differential equation

\[ y''' - y'' + y' - y = xe^x - e^{-x} + 7 \]

Example

Solve the initial value problem below subject to \( y\left(\frac{\pi}{2}\right) = -1 \) and \( y'\left(\frac{\pi}{2}\right) = 0 \).

\[ y'' + y = 8 \cos 2x \]
Important Concepts

Things to Remember from Section 4.5

1. Finding annihilators for a given $g(x)$.

2. Using annihilators to find particular solutions to a non-homogeneous linear differential equation with constant coefficients.

3. Finding the general solution to a non-homogeneous linear differential equation with constant coefficients.