MATH 312
Section 8.2: Homogeneous Linear Systems

Prof. Jonathan Duncan

Walla Walla University

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Outline

1. The Characteristic Equation
2. Distinct Eigenvalues
3. Eigenvalues With Multiplicity
4. Conclusions
The Form of a Solution

We saw in chapter 4 that many of our solutions had the form $y = c_1 e^{mx}$ where $m$ was a root of the auxiliary equation.

Solution Form

Making an assumption similar to that above, the form of a solution to a homogeneous system of differential equations $\vec{X}' = A\vec{X}$ will be

$$\vec{X} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} e^{\lambda t} = \vec{K} e^{\lambda t}$$

With this assumption in hand, we plug our “solution” into the equation to see what happens.
Expanding on our Solution

In the matrix equations below, we see the consequences of assuming that the solution vector has this form.

\[
\begin{align*}
(\vec{K}e^{\lambda t})' &= A(\vec{K}e^{\lambda t}) \\
\lambda \vec{K}e^{\lambda t} &= A(\vec{K}e^{\lambda t}) \\
\lambda \vec{K} &= A\vec{K} \\
(A - \lambda I_n)\vec{K} &= 0
\end{align*}
\]
The matrix equation $(A - \lambda I_n) \vec{K} = 0$ should look familiar to many of you.

If $\lambda$ is a scalar and $\vec{K} \neq \vec{0}$ is a column vector such that for a given $A$ the matrix equation

$$(A - \lambda I_n) \vec{K} = 0$$

is true, then $\lambda$ is called an **eigenvalue** of the matrix $A$ and $\vec{K}$ is called a corresponding **eigenvector**.

In order to find a non-trivial $\vec{K}$ to solve this equation, we must find $\lambda$ such that

$$\det (A - \lambda I_n) = 0$$

This is called the characteristic equation of the matrix $A$. 
The characteristic equation of a matrix $A$ is a polynomial equation in $\lambda$. As with any polynomial equation, we may have distinct or repeated roots.

**Three Cases**

We consider three cases which when solving characteristic equations.

- **Case I:**
  The $n \times n$ matrix $A$ possesses $n$ distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

- **Case II:**
  The $n \times n$ matrix $A$ possesses at least one real eigenvalue $\lambda$ with multiplicity greater than one.

- **Case III:**
  The $n \times n$ matrix $A$ possesses complex eigenvalues.

**Note:**

In class and on exams we will not deal with complex eigenvalues.
Dealing with Distinct Real Eigenvalues

In the case that we have $n$ distinct real eigenvalues for an $n \times n$ matrix $A$, the following theorem gives us the solution form.

**Theorem 8.7**

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be $n$ distinct real eigenvalues of the coefficient matrix $A$ for the homogeneous system $\ddot{X}' = A\ddot{X}$, and let $\vec{K}_1, \vec{K}_2, \ldots, \vec{K}_n$ be the corresponding eigenvectors. Then the general solution to the homogeneous system on the entire real line is given by:

$$\ddot{X} = c_1 \vec{K}_1 e^{\lambda_1 t} + c_2 \vec{K}_2 e^{\lambda_2 t} + \cdots + c_n \vec{K}_n e^{\lambda_n t}$$

**Solution Procedures**

Our solution procedure is thus:

- Find and solve the characteristic equation
- Find corresponding eigenvectors
- Write the solution in final form.
Examples

We apply this to several examples whose solutions will have this form.

Example

Solve the system of differential equations below.

\[
\frac{dx}{dt} = 2x + 2y \quad \quad \frac{dy}{dt} = x + 3y
\]

Example

Solve the system of differential equations below.

\[
\frac{dx}{dt} = 2x - 7y \\
\frac{dy}{dt} = 5x + 10y + 4z \\
\frac{dz}{dt} = 5y + 2z
\]
Eigenvalues with Multiplicity

If an eigenvalue has multiplicity greater than one, then we need to be more careful about our solution form.

Eigenvalues with Multiplicity

If an eigenvalue $\lambda$ has multiplicity $m < n$ there are two possibilities.

- $\lambda$ yields $m$ linearly independent eigenvectors, so that the part of the general solution corresponding to $\lambda$ is
  
  \[ c_1 \vec{K}_1 e^{\lambda t} + c_2 \vec{K}_2 e^{\lambda t} + \cdots + c_m \vec{K}_m e^{\lambda t} \]

- If there is only one eigenvector $\vec{K}$ corresponding to $\lambda$, then we have linearly independent solutions
  
  - $\vec{X}_1 = \vec{K}_{11} e^{\lambda t}$
  - $\vec{X}_2 = \vec{K}_{21} t e^{\lambda t} + \vec{K}_{22} e^{\lambda t}$, through
  - $\vec{X}_m = \vec{K}_{m1} \frac{t^{m-1}}{(m-1)!} e^{\lambda t} + \vec{K}_{m2} \frac{t^{m-2}}{(m-2)!} e^{\lambda t} + \cdots + \vec{K}_{mm} e^{\lambda t}$
An Example

We end with an example of one of these cases.

**Example**

Solve the system of differential equations below.

\[
\begin{align*}
\frac{dx}{dt} &= 6x + 5y \\
\frac{dy}{dt} &= -5x + 4y
\end{align*}
\]
Important Concepts

Things to Remember from Section 8.2

1. Finding eigenvalues and eigenvectors for a system of differential equations.

2. Solving a 1st order system with distinct eigenvalues.

3. Solving a 2nd order system with eigenvalues with multiplicity when:
   1. There are $n$ distinct eigenvectors
   2. There is only a single eigenvector