Exam II Review Sheet  
MATH 312 Spring 2008

This exam will include covered sections from 4.1-5.2 in your text. You should know general terms, definitions, and theorems from each of these sections, review the homework given for these sections, and pay particular attention to the subjects and practice problems mentioned below.

1. Existence and Uniqueness Theorem for \( n \)th-order initial-value problems.
2. The superposition principle for homogeneous equations.
3. Fundamental sets of functions and the Wronskian.
4. Initial and Boundary-Value problems.
5. Reduction of order – how to find a second solution (both formula and process).
6. Finding general solutions to \( n \)th-order linear homogeneous DEs using auxiliary equations.
7. Finding particular solutions with undetermined coefficients and annihilators.
8. Finding particular solutions to non-homogeneous linear equations by variation of parameter.
10. Solving Spring/Mass problems modeled by:
    • free undamped motion
    • free damped motion
    • driven motion
11. Eigenvalues and Eigenfunctions.

Below is a list of sample problems. This list is not all-inclusive, but does represent the types of problems you will see on the exam.

1. Does the existence-uniqueness theorem guarantee that the following initial-value problems have solutions? Justify your answers.
   (a) \( y'' + y = \ln x \) for \( y(0) = 1, y'(0) = 2 \)
   (b) \( xy'' + y = 3 \) for \( y(1) = 0, y'(1) = 1 \)
   (c) \( xy'' + y = 3 \) for \( y(0) = 0, y'(0) = 1 \)
   (d) \( y''' + y\sin x + y\cos x = \sqrt{x^2 + 1} \) for \( y(0) = 0, y'(0) = 1, y''(0) = 2 \).

2. Show that the functions \( f_1(x) = 2 \) and \( f_2(x) = x + 3 \) are linearly independent on any interval.

3. Determine whether the functions \( f_1(x) = 1, f_2(x) = \sin^2 x \) and \( f_3(x) = \cos^2 x \) are linearly independent or linearly dependent on \( (-\infty, \infty) \).

4. Consider the differential equation \( x^2 y'' - xy' + y = 0 \) for \( x \) in the interval \( (0, \infty) \).
   (a) Show that \( y_1(x) = x \) and \( y_2(x) = x \ln x \) are solutions of this differential equation.
   (b) Show that \( y_1(x) = x \) and \( y_2(x) = x \ln x \) form a fundamental set of solutions.
   (c) Give the general solution of this equation.
   (d) Find constants \( c_1 \) and \( c_2 \) so that \( y(x) = c_1 x + c_2 x \ln x \) satisfies the initial conditions \( y(1) = 3 \) and \( y'(1) = -1 \).

5. Given that \( y_1 = x \sin(\ln x) \) is a solution of \( x^2 y'' - xy' + 2y = 0 \), use reduction of order to find a second solution \( y_2 \) where \( y_1 \) and \( y_2 \) are linearly independent. Then give the general solution to the original equation. Now use the methods appropriate for the Cauchy-Euler equation to find the general solution.
6. Find general solutions to the following homogeneous equations.
   (a) \( x'' - 6x' + 9x = 0 \)
   (b) \( x'' + 9x = 0 \)
   (c) \( x^2y'' + 3xy' = 0 \)

7. Find general solutions to the following non-homogeneous equations.
   (a) \( x'' - 6x' + 9x = 10 \)
   (b) \( x'' - 4x = t \)
   (c) \( y'' + 4y' - 5y = 2e^x \)
   (d) \( x'' + 4x = \sin 2t \)
   (e) \( x^2y'' + 9xy' + 8y = x^2 \)

8. Find the solution to each of the IVPs given below.
   (a) \( y'' - 4y' - 5y = 0 \) subject to \( y(0) = 2, y'(0) = 0 \)
   (b) \( y'' + 4y = \sin t \) subject to \( y(0) = 0, y'(0) = 0 \)
   (c) \( y'' - 6y' + 8y = t^2 \) subject to \( y(1) = 0, y'(1) = 2 \)
   (d) \( y'' + y' + y = t \) subject to \( y(1) = 2, y'(1) = 0 \)

9. Find the general solution of the following ODE using variation of parameters:
   \[
   \frac{d^2y}{dx^2} + 9y = \frac{1}{4} \csc 3x
   \]

10. Given that the equation \((D^2 + x^{-1}D - x^{-2})y = 0\) has the general solution \( y = c_1x + c_2x^{-1} \), find the general solution of \((D^2 + x^{-1}D - x^{-2})y = 4x^{-1}\).

11. A 4-pound weight is attached to a spring whose spring constant is 16 lb/ft. If this weight is set in motion, what is the period of the resulting simple harmonic motion?

12. A spring is such that a 2-pound weight stretches it 6 inches. A dampening force numerically equal to the instantaneous velocity acts on the weight. The weight is pulled 3 inches below the equilibrium position and released. Find a formula for the position of the weight at time \( t \).

13. Find the eigenvalues and eigenfunctions of the boundary-value problems below.
   (a) \( y'' + \lambda y = 0 \) subject to \( y(0) = 0, y(\pi) = 0 \)
   (b) \( y'' + 2y' + (\lambda + 1)y = 0 \) subject to \( y(0) = 0, y(5) = 0 \).