Theorem: If $N \leq G$, then the map $\phi: G \to G/N$ given by $\phi(a) = aN$ is an onto homomorphism.

Proof: Onto: Since $aN \in G/N \implies a \in G$ so $\phi(a) = aN$.

To see homomorphism, note that $\phi(ab) = abN = (aN)(bN) = \phi(a)\phi(b)$.

Theorem: (First Isomorphism Theorem)

If $\phi: G \to H$ is a surjective homomorphism, with kernel $K = \ker \phi$,

then the quotient group $G/K$ is isomorphic to $H$.

Proof: Define $\psi: G/K \to H$ by $\psi(ak) = \phi(a)$ for $a \in G$.

To see that $\psi$ is well defined, let $a, b \in G$. Then, by previous work,

$\psi(ak) = \psi(bk) \iff \phi(a) = \phi(b) \iff a, b \in \ker \phi$.

Also, $\psi$ is onto, let $h \in H$. Then, as $\phi$ is surjective, there is

$a \in G$ s.t. $\phi(a) = h$. Hence, $\psi(ak) = h$ as desired. Finally,

$\psi((ak)(bk)) = \psi(ak-bk) = \phi(ab) = \phi(a)\phi(b) = \psi(ak)\psi(bk)$.

QED

Example:

Consider $\mathbb{C}^\times \setminus \{0\}$ under multiplication (a group) and $N = \{a+b|a^2+b^2=1\}$.

We claim that $N$ is a normal subgroup of $\mathbb{C}^\times$ such that $\mathbb{C}^\times/N \cong \mathbb{R}^+$, a group of positive reals under multiplication. To see this, we define a map $\phi: \mathbb{C}^\times \to \mathbb{R}^+$ with kernel $N$. Let $\phi(a+b) = a^2 + b^2$. Then, as $1 \in \mathbb{R}^+$ is the identity, $\ker \phi = N$. But $N$ is normal since multiplication in

$\mathbb{C}^\times$ is commutative. Thus, $\mathbb{C}^\times/N \cong \mathbb{R}^+$.

Theorem: If $N \leq G$ then there is a homomorphism of $G$ with kernel $N$.

Proof:

Let $\phi: G \to G/N$ be defined by $\phi(g) = gN$. This is the natural homomorphism between $G$ and $G/N$. Then $\phi$ is clearly well defined.

And $\phi(gh) = (gh)N = g(NhN) = \phi(g)\phi(h)$. Moreover, $g \in \ker \phi \iff \phi(g) = eN \iff gN = eN \iff g \in N$.\n
Abstract Algebra

what do the subgroups of $G/N$ look like for a normal $N$? for $H < G$ with $N < K$.

Theorem (Third Isomorphism Theorem)

Let $H$ be a normal subgroup of $G$, $N < K < G$. The $K/N$ is a normal subgroup of $G/N$ and the quotient group $(G/N)/(K/N)$ is isomorphic to $G/K$.

Proof: We proceed by defining a surjective homomorphism $\varphi: G/N \to G/K$, whose kernel is $K/N$. Then, we get the desired result directly from the First Isomorphism Theorem.

Note that in the group $G/N$, if $an \equiv cn$, then $a \equiv c$ in $G/K$. Thus, $ak = ck$ in $G/K$. Hence, the map $\varphi$ defined by $\varphi(an) = ak$ is well defined. Clearly $\varphi$ is onto since any $ak \in G/K$ is the image of an in $G/N$.

Furthermore, $\varphi((an)(bn)) = \varphi((ab)n) = abk = akbk = \varphi((aN)(bN))$ by def. Thus, $\varphi$ is a surjective homomorphism from $G/N$ to $G/K$. Since the identity element of $G/K$ is $e_K = K$, $\ker(\varphi) = \{a \in G : \varphi(aN) = e_K\}$ iff $a \in K$. Thus, the kernel is the set $\{aN | a \in K\}$ = $K/N$.

Since $K/N$ is a kernel of a homomorphism, it is a normal subgroup.

Thus, by the 1st isomorphism theorem $(G/N)/(K/N) \cong G/K$.

Corollary: Let $N < G$, $K \leq G$ with $N \leq K$. Then

1) $K/N$ is a subgroup of $G/N$.
2) $K/N$ is normal in $G/N$ iff $K$ is normal in $G$.
3) if $T \leq G/N$, then $T$ is a subgroup $H \leq G$ s.t. $N \leq H$ and $T = H/N$.

Proof:

1) Clearly $K/N \leq G/N$. If $aN, bN \in K/N$, then $(aN)(bN) = aN bN = ab^{-1}N \in K/N$ since $a^{-1} \in K$.

2) Let $T = G/N$. Then $K \leq N \leq K$. Let $k \in K$, $x \in G$. Then $kN \leq K/N$ as $K/N$ is normal. For $xN \in G/N$,

   $\varphi(kN) = \varphi(k)N \leq K/N$ so that $xkN \subseteq K/N$ so that $k \in G$.

3) Let $H \leq G$. Then $\ker(\varphi) = \{aN | a \in K \} = K/N$ so $\varphi(\{aN | a \in K \}) = G/K$. Hence, $H/N \leq G/N$ is normal.

Finally, $H/N = \{eN | e \in H\} \subseteq G/N$.
**Def.** A group $G$ is simple if it has only normal subgroups as itself and the trivial group.

**Prop.** Let $\varphi : G \rightarrow H$ be a surjective homomorphism, $G$ a simple group. Prove that either $\varphi$ is an isomorphism, or $H$ is the trivial group.

**pf.** By previous work, the kernel of $\varphi$ is a normal subgroup of $G$. Since the only normal subgroups of $G$ are the trivial group and $G$ itself, $\ker(\varphi) = \{e_G\}$ or $\ker(\varphi) = G$. If $\ker(\varphi) = \{e_G\}$, then by previous work, $\varphi$ is an isomorphism. On the other hand, if $\ker(\varphi) = G$, then $\forall g \in G : \varphi(g) = e_H$. Thus, $H = \{e_H\}$.

**Prop.** Let $N \triangleleft G$. Prove that $G/N$ is simple if and only if there is no normal subgroup $H \triangleleft G$ such that $N \ntriangleleft H \triangleleft G$.

**pf.** ($\Rightarrow$) Assume that $G/N$ is simple. Then $G/N$ has no proper non-trivial normal subgroups. By the previous theorem, if $H$ were as above, then $H/N$ would be a non-trivial proper normal subgroup, a contradiction. Thus, no such $H$ exists.

($\Leftarrow$) Now, by way of contraposition, assume that $G/N$ is not simple. Then, there is some $H/N$ which is normal, non-trivial, and proper. Thus, as $H/N$ is normal, $N \triangleleft H$. As $H/N$ is proper, $H \ntriangleleft G$. And as $N \triangleleft H$, $H \ntriangleleft G$.

**Hw:** Another isomorphism theorem. See textbook.