Survey of Calculus: Definite Integrals and the Fundamental Theorem

Ken Wiggins

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Definite Integral

The area under the graph of a continuous nonnegative function $f(x)$ between $x = a$ to $x = b$ is the limiting value of Riemann sums as n increases without bound or as Δx approaches zero. It can be shown that even if $f(x)$ has negative values, the Riemann sums still approach a limiting value as $\Delta x \to 0$. This number is called *definite integral of* $f(x)$ *from* a *to* b and is denoted by $\int_a^b f(x)dx$. That is,

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x].$$

Areas and Functions with Negative Values

Suppose that $f(x)$ is continuous on the interval $a \leq x \leq b$. Then

$$\int_a^b f(x)dx$$

is equal to the area above the x-axis bounded by the graph of $y = f(x)$ from $x = a$ to $x = b$ minus the area below the x-axis.

Fundamental Theorem of Calculus

Suppose that $f(x)$ is continuous on the interval $a \leq x \leq b$, and let $F(x)$ be an antiderivative of f(x). Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Examples

Find each integral.

- $$\int_0^2 \left(x^2 + x\right)\ dx$$

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- $$\int_0^2 \left(x^2 + x\right)\ dx$$
- $$\int_{-1}^1 \left(e^{2x} - 2^{-2x}\right)\ dx$$

Examples

- Find the area between the x axis and the graph of $y = -5 + 6x - x^2$.

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- Evaluate the integral

$$\int_0^2 \sqrt{4 - x^2}\, dx$$

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Plausibility argument for the Fundamental Theorem of Calculus

1. If the interval $a \leq x \leq b$ is partitioned into n subintervals of width Δx, then the net change in $F(x)$ over $a \leq x \leq b$ is the sum of the net changes in $F(x)$ over each subinterval.
2. If Δx is small, then the change in $F(x)$ over the ith subinterval is approximately $F'(x_i)\Delta x$.
3. The sum of the approximations in (2) is a Riemann sum for the definite integral $\int_a^b F'(x)dx$.

Area Function as an Antiderivative

Theorem III

Let $f(x)$ be a continuous nonnegative function for $a \leq x \leq b$. Let $A(x)$ be the area of the region under the graph of the function from a to the number x. Then $A(x)$ is an antiderivative of $f(x)$.

Examples

Find each derivative.

- $$\frac{d}{dx}\int_0^x e^{t+t^2}\,dt$$

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- $$\frac{d}{dx}\int_0^x e^{t+t^2}\,dt$$
- $$\frac{d}{dx}\int_0^{x^2+2x+2} \sqrt{t+1}\,dt.$$